Analysis Preliminary Exam: Real Analysis
January 2004

1. Give clearly reasoning and state clearly which theorems you are using.

(a) Prove that $f(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2(1+x^n)}$ is continuous for $x \geq 0$.

(b) Evaluate $\int_0^1 f(x)dx$, justifying all steps of your work, and express your answer in terms of values of the functions $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

2. Let $f(x) = \frac{\sin x}{x}$. Prove the following statements:

(a) $\int_0^{\infty} |f(x)|dx = \infty$.

(b) $\lim_{b\to\infty} \int_0^b f(x)dx = \frac{\pi}{2}$ by repeated integrating $e^{-xy}\sin x$ with respect to x and y.
Hint: you may need the formula $\int e^{au} \sin bu du = \frac{e^{au}}{a^2+b^2}(a \sin bu - b \cos bu) + C$.

3. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set and $0 < m(E) < \infty$. Prove that for every $\epsilon > 0$, there exist a set A that is a finite union of open intervals such that

$$m(E \setminus A) + m(A \setminus E) < \epsilon.$$

4. Suppose $\mu(X) < \infty$. If f and g are complex-valued measurable function on X, define

$$\rho(f,g) = \int_X \frac{|f-g|}{1+|f-g|} d\mu.$$

Prove that $\lim_{n\to\infty} \rho(f_n, f) = 0$ if and only $\lim_{n\to\infty} f_n = f$ in measure.

5. Let $(X, \mathcal{M}, \mu)$ be a positive measure space with $\mu(X) < \infty$.

(a) Show that a measurable function $f : X \to [0,\infty)$ is integrable (i.e., one has $\int_X f(x)d\mu < \infty$) if and only if the series $\sum_{n=0}^{\infty} \mu(\{x : f(x) \geq n\})$ converges.

(b) Let f be a nonnegative measurable function on [0,1] satisfying

$$m(\{x : f(x) \geq t\}) < \frac{1}{1+t^2}, \quad t > 0.$$

Using the result in (a) to determine those values of p, $1 \leq p < \infty$ for which $f \in L^p([0,1])$ and find the minimum value of p for which f may fail to be in L^p. Give an example.