Topology Qualifying Exam

Wednesday, January 7, 2004
9:00 am - 12:00 noon

1. Prove that every connected locally path connected space is path connected.

2. Let X and Y be topological spaces.
   (a) Prove that if Y is compact then the projection $p_1 : X \times Y \to X$ is closed.
   (b) Prove that if Y is compact and the graph of $f : X \to Y$ is closed in $X \times Y$, then f is continuous.

3. Let X be a compact metric space with metric d. Let $f : X \to X$ be a function. Suppose there exists a constant C such that $0 < C < 1$ and

$$d(f(x), f(y)) \leq C\, d(x, y)$$

for all $x, y \in X$. Prove that f is continuous and f has a fixed point.

4. Prove that every continuous map $f : \mathbb{R}P^2 \to S^1 \times S^1$ from the real projective plane to the torus is null-homotopic.

5. Let X be the pinched torus, which is obtained from the torus $S^1 \times S^1$ by collapsing a circle $\{x_0\} \times S^1$ to a point. Show that X is a CW complex with one 0-cell, one 1-cell, and one 2-cell. Use this cell structure to do the following:
   (a) Compute the homology groups of X.
   (b) Compute the fundamental group of X.
   (c) Find the universal covering space of X.

6. Prove that there does not exist a continuous map $f : S^2 \to S^2$ from the 2-sphere to itself such that $f(x)$ is orthogonal to x (as vectors in $\mathbb{R}^3$) for all $x \in S^2$.

7. Classify the connected 2-fold covering spaces of the Klein bottle. You may wish to use the following strategy. First classify the 2-fold covering spaces of the Möbius band. The Klein bottle is the union of two Möbius bands along their common boundary. Every covering space of the Klein bottle restricts to a covering space of each of these Möbius bands.