QUALIFYING EXAMINATION IN TOPOLOGY

January 12, 2005

Directions: Work all the problems. They are weighted evenly.

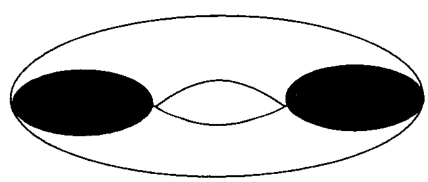
1. Suppose X is a compact metric space and $\mathcal{U}$ is an open covering of X. Prove that there is a number $\delta > 0$ so that for every $x \in X$, the ball of radius δ centered at x is contained in some element of $\mathcal{U}$.

2. Prove that a connected normal space with more than a single point must be uncountable.

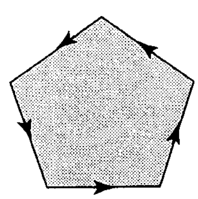
3. Suppose $f: X \to Y$ is a function, and let $G = \{(x, f(x)) : x \in X\} \subset X \times Y$.

   a. Prove that if Y is Hausdorff and f is continuous, then G is closed.
   b. Prove that if Y is compact and G is closed, then f is continuous. (Hint: Start by proving that the projection $p: X \times Y \to X$ is a closed map.)

4. Let X the topological space formed by filling in two disks in the torus, as shown. Calculate $H_*(X, \mathbb{Z})$ by any method you wish.



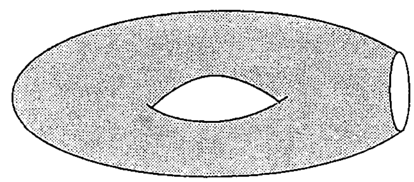
5. Let X be the topological space formed by identifying the edges of a pentagon, as shown.



Calculate $\pi_1(X)$ and give the universal covering space of X.

6. a. Suppose Y is an n-fold covering space of the (one-holed) torus. What is Y? Justify your answer.

b. Let X be the topological space formed by deleting a disk from a torus, as shown.



Suppose Y is a 3-fold covering space of X. What surfaces could Y be? Justify your answer, but you need not exhibit the covering maps explicitly.

7. Let $n \geq 2$. Prove that the covering map $\pi: S^n \to \mathbb{RP}^n$ is not nullhomotopic.

8. Define a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ by the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be endowed with the quotient topology.

a. Prove that A induces a well-defined, continuous map $f: T^2 \to T^2$.

b. Suppose $g: T^2 \to T^2$ is a continuous map that is homotopic to f. Prove or disprove: g has a fixed point.

9. Do one of the following.

a. Give (with justification) a contractible subset $X \subset \mathbb{R}^2$ that is not a retract of $\mathbb{R}^2$.

b. Give (with justification) two topological spaces that have the same homology groups but that are not homotopy equivalent.