Topology Qualification Exam, Spring 2024

Instructions: You can assume homology groups and fundamental groups of a point and wedges of spheres in all dimensions. Everything else should be computed.

1. (a) Show that, if $X$ is a Hausdorff space and if $A$ is a subset of $X$ that is compact with respect to the subspace topology, then $A$ is closed as a subset of $X$.
   (b) Give an example showing that part (a) would no longer be true if the Hausdorff assumption on $A$ were dropped.

2. Let $(X, d)$ be a metric space and let $\mathcal{B} = \{B_\alpha\}_{\alpha \in A}$ be a collection of nonempty open subsets that is a base for the topology on $X$. For each $\alpha$, let $x_\alpha \in B_\alpha$. Prove that $\{x_\alpha\}_{\alpha \in A}$ is a dense subset of $X$.

3. Let $A$ and $B$ be two Möbius strips, and let $X$ be the space formed by gluing $A$ and $B$ together by a homeomorphism between their boundary circles.
   (a) Compute the fundamental group of $X$.
   (b) Compute all homology groups of $X$.
   (c) Identify $X$ in terms of the classification of surfaces.
   (d) Find a connected 2-sheeted covering space for $X$.

4. Prove that if $X$ is a topological space and $A$ is a subset such that $A$ has more path components than $X$ does, then the relative homology $H_1(X, A)$ is nonzero.

5. Let $X$ be the topological space obtained from $\mathbb{R}^3$ by removing $x$-, $y$- and $z$-axis. Compute the fundamental group of $X$.

6. (a) Let $\rho_3 : S^1 \to S^1$ be the $2\pi/3$-rotation, and $X_3$ be the topological space obtained from $[0, 1] \times S^1$ by identifying each $(1, x)$ with $(1, \rho_3(x))$ for all $x \in S^1$. Compute $\pi_1(X_3)$.
   (b) Let $Y$ be the topological space obtained from attaching $X_3$ to $S^1 \times S^1$ by identifying $\{0\} \times S^1$ with $\{x\} \times S^1$ via the identity map. Compute all homology groups of $Y$.

7. Determine whether the following statements are true or false. Prove it if it is true, and find a counter example if it is false.
   (a) For $n > 1$, every continuous map from $S^n$ to $T^n = S^1 \times S^1 \times \cdots \times S^1$ is nullhomotopic.
   (b) For $n > 1$, every continuous map from $T^n$ to $S^n$ is nullhomotopic.

8. Compute all homology groups of $S^1 \times \mathbb{R}P^2$ using cellular homology.