Topology Qualifying Exam Fall 2019
Fall 2019

Work on the **first two** and **6 of the last 7 problems**. Give clear explanations and complete proofs. If you work on all 7 of the last 7 problems, **be sure to cross out the one you do not want graded**, otherwise the first 6 to appear will be graded.

(1) (a) Prove that the union of a collection of connected subspaces of a topological space X that have a point in common is connected.

(b) Let X and Y be connected spaces. Prove that $X \times Y$ endowed with the product topology is connected.

(2) Let $f : X \to Y$ be a closed, continuous, surjective map such that $f^{-1}(\{y\})$ is compact, for each $y \in Y$. Prove that if Y is compact then X is compact.
Hint: You may want to start the proof by showing that if U is an open set with $f^{-1}(\{y\}) \subset U$, then there exist a neighborhood W_y of y such that $f^{-1}(W_y) \subset U$.

(3) Let $T^2 = S^1 \times S^1$ be the torus, represented by a labelled rectangle in the usual way. Let S be the sub-complex represented by the top edge of the rectangle (the longitude of the torus). Compute the relative homology groups $H_*(T^2, S; \mathbb{Z})$.

(4) Let X be the space constructed by identifying the boundary of a Möbius band M with longitude of the torus $T^2 = S^1 \times S^1$. Compute the fundamental group $\pi_1(X)$ and the homology groups $H_*(X, \mathbb{Z})$.

(5) Define the suspension $S(X)$ to be the space obtained from $X \times [0, 1]$ by contracting $X \times \{0\}$ to a point S and contracting $X \times \{1\}$ to another point N.

(a) Let Y be a disjoint union of two circles. Compute the integer homology groups of $S(Y)$.

(b) More generally, for a topological space X, describe the (integer) homology groups of $S(X)$ in terms of the homology groups of X.

(6) Let Σ_g denote a closed surface of genus g, and $\Sigma_{g,k}$ a surface of genus g with k boundary components (i.e. Σ_g with k disks removed). Which of the surfaces $\Sigma_{g,k}$ support a map homotopic to the identity that does not have a fixed point?

(7) Find all (connected and disconnected) 3-fold covers of the wedge of a circle and the projective plane, $\mathbb{R}P^2 \vee S^1$. Justify your answer.

(8) Show that any continuous map $f : \mathbb{R}P^2 \to S^1 \times S^1$ is null homotopic.

(9) Does there exist a map of degree 2019 between $\mathbb{S}^3$ and $\mathbb{S}^3$?