Topology Qualifying Exam, January 2019

(1) Is every complete bounded metric space compact? If so, give a proof; if not, give a counterexample.

(2) Let X be a Hausdorff topological space. Recall that the **one-point compactification** $\bar{X}$ of X is given by the following.

- As a set $\bar{X} = X \cup \{\infty\}$, where ∞ is a point not belonging to X.
- A subset of $\bar{X}$ declared to be open if either it is an open subset of X, or it is of the form $U \cup \{\infty\}$, where $U \subset X$ and $X - U$ is compact.

Prove that the above description of open sets defines a topology on $\bar{X}$, and that $\bar{X}$ is compact under this topology.

(3) Let $p : X \to Y$ be a covering space, where X is compact, path-connected, and locally path-connected. Prove that for each $x \in X$ the set $p^{-1}(\{p(x)\})$ is finite, and has cardinality equal to the index of $p_*(\pi_1(X, x))$ in $\pi_1(Y, p(x))$.

(4) Is there a covering map from

$$X_3 := \{x^2 + y^2 = 1\} \cup \{(x-2)^2 + y^2 = 1\} \cup \{(x+2)^2 + y^2 = 1\} \subset \mathbb{R}^2$$

to the wedge of two S^1's? If there is, give an example; if not, give a proof.

(5) (i) Consider the quotient space $\mathbb{T}^2 = \mathbb{R}^2/\sim$, where $(x, y) \sim (x+m, y+n)$ for $m, n \in \mathbb{Z}$, and let A be any 2×2 matrix whose entries are integers such that $\det A = 1$. **Prove** that the action of A on $\mathbb{R}^2$ descends via the quotient $\mathbb{R}^2 \to \mathbb{T}^2$ to induce a homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$.

(ii) Using this homeomorphism of $\mathbb{T}^2$, we define a new quotient space $T^3_A := \frac{\mathbb{T}^2 \times \mathbb{R}}{\sim}$, where $((x, y), t) \sim (A \cdot (x, y), t+1)$. **Compute** $H_1(T^3_A)$ in the case that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(6) (i) Use the Lefschetz fixed point theorem to show that any degree-one map from $f : S^2 \to S^2$ has at least one fixed point.

(ii) Give an example of a map $\mathbb{R}^2 \to \mathbb{R}^2$ having no fixed points.

(iii) Give an example of degree-one map $S^2 \to S^2$ that has only one fixed point.

(7) For topological spaces X, Y, the **mapping cone** $C(f)$ of a map $f : X \to Y$ is defined as $(X \times [0, 1]) \sqcup Y/\sim$, where $(x, 0) \sim (x', 0)$ and $(x, 1) \sim f(x)$. Let $\phi_k : S^1 \to S^1$ be a k-fold covering. Find $\pi_1(C(\phi_k))$.

(8) Let Σ be a connected compact surface and let $p_1, \cdots, p_k \in \Sigma$. Prove that $H_2(\Sigma - \cup_{i=1}^k \{p_i\}) = 0$.