Topology Qualifying Exam
August 15, 2006
8:30–11:30 am

In problems 1, 2, and 3, give complete arguments starting from the definitions.

1. A topological space X is *sequentially compact* if every infinite sequence in X has a convergent subsequence. Prove that every compact metric space is sequentially compact.

2. (a) Prove that if the space X is connected and locally path connected then X is path connected.
   (b) Is the converse true? Give a proof or a counterexample.

3. If f is a function from X to Y, consider the graph $G = \{(x, y) \in X \times Y \mid f(x) = y\}$.
   (a) Prove that if f is continuous and Y is Hausdorff, then G is a closed subset of $X \times Y$.
   (b) Prove that if G is closed and Y is compact, then f is continuous.

4. (a) State van Kampen's theorem.
   (b) What are the fundamental groups of the torus T^2 and the real projective plane $\mathbb{R}P^2$? (Do not prove your answers.)
   (c) Use van Kampen's theorem and part (b) to compute the fundamental group of the one-point union $X = T^2 \vee \mathbb{R}P^2$.

5. (a) Give the definitions of *covering space* and *deck transformation* (or *covering transformation*).
   (b) Describe the universal cover of the Klein bottle and its group of deck transformations.

6. Prove that there does not exist a continuous map $f : S^2 \to S^2$ from the unit sphere in $\mathbb{R}^3$ to itself such that $f(x) \perp x$ (as vectors in $\mathbb{R}^3$) for all $x \in S^2$.

7. (a) What is the degree of the antipodal map $a_n : S^n \to S^n$? (Do not prove your answer.)
   (b) Define the structure of a CW complex on real projective n-space $\mathbb{R}P^n$.
   (c) Use parts (a) and (b) to compute the cellular homology of $\mathbb{R}P^n$. Show your work.

8. (a) State the Mayer-Vietoris theorem.
   (b) Use it to compute the homology of the space X obtained by gluing two solid tori along their boundary as follows. Let D^2 be the unit disk and let S^1 be the unit circle in the complex plane $\mathbb{C}$. Let $A = S^1 \times D^2$ and $B = S^1 \times D^2$. Then X is the quotient space of the disjoint union $A \sqcup B$ obtained by identifying $(z, w) \in A$ with $(zw^3, w) \in B$ for all $(z, w) \in S^1 \times S^1$.