Topology Qualifying Exam, January 2017

(1) Compute $H_0(X)$ where X is shown in Figure 1.

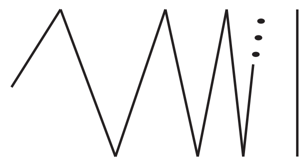

Figure 1

(2) Let Y be the annulus with identifications as shown in Figure 2.

(a) Explain why Y is a surface.

(b) Is Y orientable?

(c) What surface is Y?

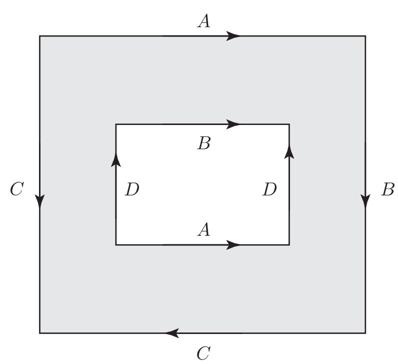

Figure 2

(3) Show that $S^1 \times S^1$ is not the union of two disks (where there is no assumption that the disks intersect along their boundaries).

(4) Suppose that a continuous map $f : S^3 \times S^3 \to \mathbb{R}P^3$ is not surjective. Prove that it is homotopic to a constant function.

(5) (a) Show that any finite index subgroup of a finitely generated free group is free. State clearly any facts you use about fundamental groups of graphs.

(b) Prove that if N is a nontrivial normal subgroup of infinite index in a finitely generated free group F, then N is not finitely generated.

(6) Find all three-fold covers of the wedge of two copies of $\mathbb{R}P^2$. Justify your answer.

(7) Let $X = S_1 \cup S_2 \subset \mathbb{R}^3$ be the union of two spheres of radius 2, one about $(1,0,0)$ and the other about $(-1,0,0)$, i.e. $S_1 = \{(x,y,z)|(x-1)^2 + y^2 + z^2 = 4\}$ and $S_2 = \{(x,y,z)|(x+1)^2 + y^2 + z^2 = 4\}$.
a) Give a description of X as a CW complex.
b) Write out the cellular chain complex of X.
c) Calculate $H_*(X, \mathbb{Z})$.

(8) Use the circle along which the connected sum is performed and the Mayer-Vietoris long exact sequence to compute the homology of $RP^2 \# RP^2$.

(9) Prove or disprove. Every map from $\mathbb{R}P^2 \vee \mathbb{R}P^2$ to itself has a fixed point.