Complex analysis qualifying examination, Fall 2009

Do all problems, and justify your assertions.

Problem 1. *Let f be analytic on the unit disk. Is it possible that $f(\frac{1}{n})$ takes the following values for $n = 2, 3, 4, \ldots$? Why or why not?*

a) $(-1)^n$

b) e^{-n} *for n even, and 0 for n odd*

c) $\frac{1}{\lfloor \sqrt{n} \rfloor}$ *(here $\lfloor x \rfloor$ is the greatest integer not greater than x)*

d) $\frac{n-2}{n-1}$

Problem 2. *Use complex analysis to prove the Fundamental Theorem of Algebra: if $p(z)$ is a non-constant polynomial with complex coefficients, then there exists a complex number z_0 such that $p(z_0) = 0$.*

Problem 3. *Compute the integral*

$$\int_0^\infty \frac{\cos x}{(1 + x^2)^2}\, dx$$

Problem 4. *Give the Laurent series for the following functions, and characterize their singularities at zero (essential, removable or pole):*

a) $\frac{\sin^2 z}{z}$

b) $ze^{-\frac{1}{z^2}}$

c) $\frac{1}{z(4-z)}$

Problem 5. *Let D be the domain in the complex plane consisting of the interior of the unit disk, with the subdisk of radius $\frac{1}{2}$ and center $\frac{1}{2}$ removed. Construct an analytic function that maps D conformally onto the first quadrant $\Re z > 0, \Im z > 0$. (See figure.)*

Problem 6. *Give Laplace's equation in polar coordinates. In other words, if $x = r\cos\theta, y = r\sin\theta$, write down a relation equivalent to*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

in terms r, θ, and the partial derivatives of f with respect to r, θ.