Complex Analysis Qualifying Exam — Fall 2022

Show work and carefully justify/prove your assertions. For example, if you use a theorem that has a name, mention the name. Arrange your solutions in numerical order even if you do not solve them in that order.

1. (10 points) Consider the $n-1$ diagonals of a regular n-gon inscribed in a unit circle obtained by connecting one vertex with all the others. Use complex numbers to show that the product of their lengths is n. [Hint: Let $z_k \neq 1$ be roots of $z^n - 1 = 0$, $k \leq n-1$. Then $|z_k - 1|$ is the length of a diagonal.]

2. (10 points) Use complex analysis to compute
$I = \displaystyle\int_0^\infty \frac{x^\alpha}{1+x^2} dx, \quad \text{where } -1 < \alpha < 1.$

3. (10 points) Let $f(z)$ be a complex valued function. Show that if both $f(z)$ and $zf(z)$ are harmonic in a domain Ω, then f is holomorphic in Ω.

4. (10 points) Let $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ have the radius of convergence r and assume the function $f(z)$ have exactly a simple pole at z_0 with $|z_0| = r$. Prove that (i) $a_n \neq 0$ for all large enough n's, and (ii) $\displaystyle\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$.

   Hint: Write $f(z) = \dfrac{A}{z - z_0} + g(z)$, where A is the residue of $f(z)$ at z_0 and $g(z)$ is a power series in z.

5. (7 + 13 points) Let $f(z) = \dfrac{\pi^2}{\sin^2 \pi z} - \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

   (i) Show that $f(z)$ has only removable singularities on the complex plane (therefore it can be viewed as an entire function).

   (ii) Show that $f(z) = 0$ if $0 \leq \text{Re}(z) \leq 1$, and conclude that the following identity hold on the complex plane
   $$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

6. (10 + 10 points) (i) Show that if f is a one-to-one holomorphic map from the unit disc $\mathbb{D}$ onto the upper half plane $\mathbb{H}$, then f is a fractional linear transformation.

   (ii) Find the most general form of all f in part (1) as explicitly as possible.