5200/7200 Metric geometry Axioms.
We are given a set S of points, a collection of subsets of S called lines, and a symmetric, non negative distance function on pairs of points of S, such that two points have distance zero apart if and only if they are equal.

Incidence: i) There exist at least three points of S that do not lie on the same line.
ii) Every pair of distinct points of S do lie on a unique line.
iii) Every line contains at least two distinct points.

Rulers: Every line has a "ruler", (distance preserving bijection to the real numbers).

Plane separation: Given a line L, define a relation on S-L (the complement of L in S) by saying x and y are related iff the segment xy does not meet L. Then this relation is an equivalence relation with exactly two non empty (convex) equivalence classes.

Protractors: There is an angle measure function from all non straight convex angles to the interval (0,180), which is additive, and such that: given a fixed ray from p, and a side of the corresponding line L, angle measure is a 1-1 correspondence between all rays from p lying in the given side of L, and the interval (0,180).

Rigid motions: (1-1 correspondences S→S, preserving lines, distances, angle measure)
i) For each pair p,q of points of S, there is a rigid motion taking p to q.
ii) For each pair of rays A,B at p, there is a rigid motion taking A to B, and fixing p.
iii) For each line L, there is a rigid motion fixing all points of L, and interchanging sides of L.

Up to here these axioms define a **neutral geometry.**

Two distinct lines satisfy the Z principle, if for any transversal, the alternate angles on the same side of the transversal are equal, or equivalently if the two interior angles on the same side of the transversal add up to a straight angle.

We define lines to be parallel if they do not meet.

Euclid's fifth postulate (EFP): Two parallel lines always satisfy the Z principle.
A neutral geometry in which EFP hold is called a Euclidean geometry.

Theorems about triangles
We begin with theorems which do not require the assumption of the EFP, and hence which hold in every neutral geometry.

Congruence of segments and angles means equality of length or angle measure.
Congruence of triangles is a correspondence between vertices inducing congruence of all corresponding sides and angles.

Theorem: A distance preserving map taking p to x and q to y, takes segment pq to segment xy.
Proof: Segment pq consists of points collinear with p,q and lying between them. A rigid motion preserves collinearity and distance, and since betweenness is defined in terms of distance, it also preserves betweenness. Hence it takes those points between p and q to those points between x and y. **QED.**

Exercise: If a distance preserving map takes p to q, it takes a ray A at p, to a ray X at q, hence takes an angle AB with vertex p, to an angle XY with vertex q.

Theorem: A distance preserving map taking line L to line M, takes all points on the same side of L to points on the same side of M.
Proof: If p→x and q→y, then segment pq goes to segment xy. If p,q are on the same side of L, the segment pq does not meet L, so its image, segment xy, does not meet M, hence xy are on the same side of M. **QED.**

Theorem: A distance preserving map taking p to q, and rays A,B at p, to rays X,Y at q, takes the inside of angle AB to the inside of angle XY.
Proof: If r = a point of ray A, and s = a point of ray B, then the interior of angle AB, is those points on the same side of the line through A as are the points of B, and also on the same side of the line through B as the points of ray A. Hence by the previous result, a motion preserving the property of points being on the same side of a line, also preserves the property of points being inside an angle. **QED.**

Now we get the basic principle for showing congruence of triangles by rigid motions.

Thm: If A,B,C are non collinear points, and there is a rigid motion taking A,B,C to X,Y,Z, then triangles ABC and XYZ are congruent. In particular, side AB = side XY, side BC = side YZ, side CA = side ZX, and angles A,B,C = angles X,Y,Z respectively.
Proof: We know the motion takes segments of a certain length to segments of that same length, hence the sides of ABC are congruent to corresponding sides of XYZ. Moreover, it takes the angles formed by the pairs of sides of ABC, the angles formed by the pairs of sides of XYZ, preserving measure. Hence all the angles of ABC are congruent to the corresponding ones of XYZ. **QED.**

The point here is that you only need to see that the vertices of the triangles correspond under a rigid motion, and the sides and angles take care of themselves.

The following nice proof for dropping a perpendicular, was shown to me by Katie.
Theorem: For any line L and any point p off L, there is at least one line through p perpendicular to L.
Proof: By existence of rigid motions, there is a rigid motion (reflection in L), leaving points of L fixed and sending points from one side of L to points on the other side. Let p go to q under such a motion, and connect p to q by a segment meeting L at x. We claim that line pq is perpendicular to L at x. Let y be another point of L away from x. Since

the reflection sends x to x and p to q, it sends the segment px to the segment qx, and sends the angle pxy, to the angle qxy, whence these angles are equal. But when aded together these two angles add up to the straight angle pxq. Hence both angles pxy and qxy must equal 90 degrees. **QED.**

We will see soon that there cannot be more than one perpendicular to l from the point p.

More criteria for checking congruence of triangles.
Congruence of all 3 sides and all 3 angles, is 6 pieces of information. We want to check that we can get by on much less, usually just three pieces of that information - in fact not quite any three pieces, but most triples will do.

Theorem(SAS): If ABC and XYZ are triangles, such that side AB = side XY, side BC = side YZ, and angle B = angle Y, then the triangles are congruent, i.e. side CA = side ZX, angle A = angle X, and angle C = angle Z.
Proof: By the assumptions for rigid motions, there is a rigid motion taking side AB onto side XY, and vertex C to the side opposite from vertex Z of line XY. Then reflection in line XY, takes side BC onto side XY, because of the equalities of angle A = angle X, and side AB = side XY. Now we have taken A to X, B to Y, and C to Z, so we are done, by the basic principle of rigid motions and triangle congruence. **QED.**

Theorem (Isosceles triangle principle): A triangle with two equal sides has the angles opposite those sides also equal.
Proof: Consider the triangle ABC, and the triangle XYZ, with the same vertices, but taken in a different order. I.e. if sideAB = side BC, let X = C, and Z = A, and Y = B. Then by hypothesis side AB = side, and side BC = side YZ. Moreover angle B = angle Y, so this correspondence between triangles is a congruence, with angle A corresponding to angle X, hence those angles are equal. I.e. angle A = angle X = angle C. **QED.**

Because this proof involves a congruence between ABC and itself, it is slightly confusing psychologically. Thus Euclid gave a diferent proof using aditional triangles that are easier to visualize, since they are different.

Theorem(SSS): If two triangles ABC, and XYZ have sideAB=sideXY, side BC = side YZ, and side ZX =side CA, then the triangles are congruent by this correspondence, i.e. angle A=angleX, angleB=angleY, and angleC=angleZ.
Proof: We try to find a rigid motion taking ABC to XYZ. We can find one taking side AB to side XY, and taking C to a point C’ on the side opposite from Z of the line XY. Then consider the triangle with vertices XZC’. Since side XZ = side XC’ (why?) this is isosceles,and angle XZC’ = angle XC’Z.
Repeating this argument for triangle YZC’, we combine these results to conclude that angle XZY=angleXC’Y. (Draw a picture.) Hence by SAS the triangles are congruent as claimed. **QED.**

Theorem(ASA): If triangles ABC, and XYZ, have angle A = angleX, angleB = angleY, and sideAB=sideXY, then they are congruent, i.e. sideBC=sideYZ, sideCA=sideZX, and angleC=angleZ.
Proof: We can find a rigid motion taking side AB to sideXY, and point C to a point C' on opposite side of line XY from point Z, as usual. Now reflecting in line XY sends side XC' up to a segment making the same angle as side XZ, hence C' goes to a point C'' lying on the line through XZ. Similarly segment C'Y goes to a segment making the same angle with XY as does side YZ, so the point C'' that C' goes to, is also on the line through YZ. Now since C'' is on both lines XZ and YZ, it must be that C'' = Z. Since we have found a rigid motion taking A,B,C to X,Y,Z, we are done. **QED.**

There is one more congruence theorem, "AAS", that is a little harder to prove, but it is nicely proved in Euclid, and we will follow his line of argument. It starts with a theorem I was not too familiar with, the "exterior angle theorem". This and the next few results are inequalities rather than equalities, and hence have a slightly new flavor. Recall that Euclid stated a single principle of inequality, that "the whole is greater than the part". Hence we will prove all our inequalities by arranging for the smaller quantity to be a part of the larger. First however we prove a result Euclid overlooked, presumably as obvious.

Theorem(Pasch): If a line L does not meet any vertices of a triangle, then it meets an even number of sides of the triangle, i.e. either zero or two of the sides.
Proof: If L does not meet triangle ABC at all, the conclusion holds, so assume L meets side AB somewhere between A and B. This means that A and B are on opposite sides of L. Now C must be on the same side as either A or B, but not both, hence exactly one of the segments AC or BC also meets L. **QED.**

Exercise: Show that no line can meet only one side of a closed polygon.

Definition: A polygon is called convex, if no three vertices are collinear, and if for every line L containing the two endpoints of some side, all vertices not on L are on the same side of L. Hence the line through two adjacent vertices of a convex polygon determines a unique halfplane, the one containing the other vertices. The interior of a convex polygon is the intersection of these halfplanes.

Exercise: A distance preserving map takes a convex polygon to a convex polygon.

Remark: It seems also that a line missing the vertices of a polygon, always meets an even number of sides, and if the polygon is convex, it meets either zero or two sides. Can you prove this?

Remark: The "interior" of a general polygon P seems to be describable as follows: given a point x not a vertex of P, choose any line L containing x but not meeting any vertices of P. Then L contains an even finite number of points of P, at most equal to the number of sides of P. If an odd number of these points of L are on one side of x, and hence also an odd number are on the other side of x, then x is an interior point of P. If on the other hand an even number of these points are on each side of x, then x is an exterior point of P. It seems a non trivial theorem that any two interior points of P can be connected by a chain of segments which do not meet P, and the same is true for any two exterior points of P. (Remember P could have say, one trillion sides.)

The next result is Proposition 16, Book one, in Euclid.

Theorem(exterior angle theorem): Given a triangle ABC, extend one side, say AB, past vertex B, to some point E, forming a new angle EBC, called the exterior angle, at B. Then angle EBC is greater than either of the other two angles of the triangle, i.e. greater than angle ACB and angle CAB.

Note: Since angleEBC= 180-angleABC, this theorem equivalent to saying that neither of the other two angles can add up to as much as 180 with angle ABC, i.e. that the sum of any two angles in a triangle is always less than 180.

Proof: The first step of the proof will be to construct an angle inside angle EBC, which is equal to angle ACB. To do it Euclid marks the midpoint X of side BC, and constructs segment AX, continuing it on the other side of BC to a point Y, with segment AX = segment XY. Then by SAS, triangles ACX and YBX are congruent, in that order, hence angle YBC = angleYBX = angle ACX = angle ACB. [Euclid essentially quits here. But we press on, with a closer look at the topological aspects of the argument.]

We will be done if we see that angle YBC is inside of angle EBC. This means point Y is interior to angle EBC, which means it is on the same side of BC as point E, and on the same side of BE as point C. But A and Y are on opposite sides of lineBC, since AY meets BC at X. And also A and E are on opposite sides of BC. Hence Y and E are on the same side. Similarly, one sees that C and X are on the same side of AE, as are Y and X, so C and Y are on the same side of line AE.

Now to show that angle EBC is also larger than angleCAB, it suffices to use the same construction to show instead that the vertical angle with angleEBC is larger, since vertical angles are equal. **QED.**

Corollary: Two lines with a common perpendicular do not meet.
Proof: If they did the resulting triangle would have two 90 degree angles, violating the previous result. **QED**.

Corollary: Two lines which meet do not satisfy the Z principle for transversals, i.e. the alternate angles on one side of a transversal are never equal.
Proof: One of the alternate angles is the exterior angle of a triangle for which the other is a remote interior angle. **QED.**

Corollary: If two lines do satisfy the Z principle for some transversal, then the two lines are parallel.
Proof: Contrapositive of previous result. **QED.**

Corollary: If p is a point off the line L, there is exactly one line through p which is perpendicular to L.
Proof: We know there is at least one line perpendicular to p. If there were two such, these two perpendiculars to L would meet at p, contradicting the previous result. **QED.**

Corollary: Through a point p off a line L, there passes at least one line parallel to L.
Proof: Let M be a line through p perpendicular to L. Then construct a line K through p perpendicular to M. Since both L and K are perpendicular to M, they are parallel.

Next we prove the AAS criterion for congruence of triangles.

Theorem(AAS): In in triangles ABC, and XYZ we have sideBC = side YZ, and angle A = angleX, angleB =angleY, then the triangles are congruent in this ordering.
Proof: We will show that knowing the length of side BC and angles A and B, also determines the length of side AB. For if not, and we can extend side BA past vertex A, to become longer, say to point D, then applying the exterior angle principle to angleBAC exterior to triangle CAD, shows the angle ADC cannot be as large as angle BAC. **QED.**

[This is a short version of the proof, because I am getting tired.]

We want to prove the triangle inequality, that two sides of a triangle, added together, always exceed the length of the third side. This is Euclid's proposition 20, and its proof uses one more principle of inequality, Propositions 18,19, which say that in any triangle the larger of two angles has the longer side opposite to it.

Theorem: In triangle abc, if side ac is longer than side ab, then angle abc > angle acb.
Proof: Since side ac> side ab, we may draw a point b' between a and c, such that d(a,b') = d(a,b), and may extend side ab to a point c' with d(a,c') = d(a,c). Then triangles abc and ab'c' are congruent by SAS. Hence exterior angle abc > angle ac'c > angle ac'b' = angle acb, as claimed. **QED.**

Theorem(triangle ≠): In any triangle the sum of the lengths of any two sides is greater than the length of the third side.
Proof: In triangle abc, extend side ab to a point c', such that d(b,c') = d(b,c). Then triangle bcc' is isosceles, so angle ac'c is congruent to a part of angle acc', hence is smaller. Then the side opposite angle ac'c, namely side ac, is shorter than the side opposite angle acc', namely the side ac', whose length is the sum of the lengths of sides ab and bc. **QED.**

We can also prove that two distinct circles cannot meet more than twice, i.e. three points lie on at most one circle, and that two intersecting lines determine equal "vertical angles.

Although Euclid did not do so, it is possible to continue with a few more results that do not use the parallel postulate, and these are due to the Italian Jesuit mathematician Saccheri. The principal result is that the angle sum in a triangle is never more than 180 degrees. Thus not only is the sum of any two angles of a triangle always less than 180, but even the sum of all three angles is no more than 180. It seems tempting to try to deduce this slightly stronger result from the previous one, but I do not see how. Do you?

We introduce the concept of a Saccheri quadrilateral, and prove that the top edge is always at least as long as the base edge, and the top angles are no greater than 90 degrees. Then we deduce also that the sum of the angles of a triangle is no more than 180. We also prove that a right triangle is determined by any two side lengths.

Finally we move on to Euclidean geometry, i.e. we assume EFP. We get then that every Saccheri quadrilateral is a rectangle, that Pythagoras holds, that triangles have angle sum actually = 180 degrees, and through a point p off a line L, there is exactly one line M parallel to L. Moreover, two distinct lines have a common perpendicular iff they are parallel iff they have a constant distance apart iff they satisfy the Z principle.