Algebra Qualifying Examination, Spring 2018

Justify all the calculations and state the theorems you use in your answers. Each problem is worth 10 points.

1. (a) Use the Class Equation (equivalently, the conjugation action of a group on itself) to prove that any p-group (a group whose order is a positive power of a prime integer p) has a nontrivial center.
   (b) Prove that any group of order p^2 (where p is prime) is abelian.
   (c) Prove that any group of order $5^2 \cdot 7^2$ is abelian.
   (d) Write down exactly one representative in each isomorphism class of groups of order $5^2 \cdot 7^2$.

2. Let $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$.
   (a) Find the splitting field K of f, and compute $[K : \mathbb{Q}]$.
   (b) Find the Galois group G of f, both as an explicit group of automorphisms, and as a familiar abstract group to which it is isomorphic.
   (c) Exhibit explicitly the correspondence between subgroups of G and intermediate fields between $\mathbb{Q}$ and K.

3. Let K be a Galois extension of $\mathbb{Q}$ with Galois group G, and let E_1, E_2 be intermediate fields of K which are the splitting fields of irreducible $f_i(x) \in \mathbb{Q}[x]$. Let $E = E_1E_2 \subset K$. Let $H_i = \mathrm{Gal}(K/E_i)$ and $H = \mathrm{Gal}(K/E)$.
   (a) Show that $H = H_1 \cap H_2$.
   (b) Show that H_1H_2 is a subgroup of G.
   (c) Show that $\mathrm{Gal}(K/(E_1 \cap E_2)) = H_1H_2$.

4. Let
$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 1 & -3 \\ 1 & 2 & -4 \end{bmatrix} \in M_3(\mathbb{C})$$
   (a) Find the Jordan canonical form J of A.
   (b) Find an invertible matrix P such that $P^{-1}AP = J$. (You should not need to compute P^{-1}.)

5. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N = \begin{pmatrix} x & u \\ -y & -v \end{pmatrix}$ over a commutative ring R, where b and x are units of R. Prove that if $MN = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$, then $MN = 0$.

6. Let $M = \{(w,x,y,z) \in \mathbb{Z}^4 \mid w + x + y + z \in 2\mathbb{Z}\}$, and $N = \{(w,x,y,z) \in \mathbb{Z}^4 \mid 4|(w-x), 4|(x-y), 4|(y-z)\}$.
   (a) Show that N is a $\mathbb{Z}$-submodule of M.
   (b) Find vectors $u_1, u_2, u_3, u_4 \in \mathbb{Z}^4$ and integers d_1, d_2, d_3, d_4 such that $\{u_1, u_2, u_3, u_4\}$ is a free basis for M, and $\{d_1u_1, d_2u_2, d_3u_3, d_4u_4\}$ is a free basis for N.
   (c) Find vectors $u_1, u_2, u_3, u_4 \in \mathbb{Z}^4$ and integers d_1, d_2, d_3, d_4 such that $\{u_1, u_2, u_3, u_4\}$ is a free basis for M, and $\{d_1u_1, d_2u_2, d_3u_3, d_4u_4\}$ is a free basis for N.
   (d) Use the previous part to describe M/N as a direct sum of cyclic $\mathbb{Z}$-modules.

7. Let R be a PID and M be an R-module. Let p be a prime element of R. The module M is called $\langle p \rangle$-primary if for every $m \in M$ there exists $k > 0$ such that $p^k m = 0$.

   (a) Suppose M is $\langle p \rangle$-primary. Show that if $m \in M$ and $t \in R$, $t \notin \langle p \rangle$, then there exists $a \in R$ such that $atm = m$.

   (b) A submodule S of M is said to be *pure* if $S \cap rM = rS$ for all $r \in R$. Show that if M is $\langle p \rangle$-primary, then S is pure if and only if $S \cap p^k M = p^k S$ for all $k \geq 0$.

8. Let $R = C[0,1]$ be the ring of continuous real-valued functions on the interval $[0,1]$. Let I be an ideal of R.

   (a) Show that if $f \in I$, $a \in [0,1]$ are such that $f(a) \neq 0$, then there exists $g \in I$ such that $g(x) \geq 0$ for all $x \in [0,1]$, and $g(x) > 0$ for all x in some open neighborhood of a.

   (b) If $I \neq R$, show that the set $Z_I = \{x \in [0,1] \mid f(x) = 0, \text{ all } f \in I\}$ is nonempty.

   (c) Show that if I is maximal, then there exists $x_0 \in [0,1]$ such that $I = \{f \in R \mid f(x_0) = 0\}$.