Algebra Qualifying Examination, August 2018

Justify all the calculations and state the theorems you use in your answers. Each problem is worth 10 points. In the solution of a part of a problem, you may use any earlier part of that problem, whether or not you've correctly solved it.

1. Let G be a finite group whose order is divisible by a prime number p. Let P be a normal p-subgroup of G (so $|P| = p^c$ for some c).
   (a) Show that P is contained in every Sylow p-subgroup of G.
   (b) Let M be a maximal proper subgroup of G. Show that either $P \subseteq M$ or $|G/M| = p^b$ for some $b \leq c$.

2. (a) Suppose the group G acts on the set X. Show that the stabilizers of elements in the same orbit are conjugate.
   (b) Let G be a finite group and let H be a proper subgroup. Show that the union of the conjugates of H is strictly smaller than G, i.e.
   $$\bigcup_{g \in G} gHg^{-1} \subsetneq G.$$
   (c) Suppose G is a finite group acting transitively on a set S with at least 2 elements. Show that there is an element of G with no fixed points in S.

3. Let $F \subset K \subset L$ be finite degree field extensions. For each of the following assertions, give a proof or a counterexample.
   (a) If L/F is Galois, then so is K/F.
   (b) If L/F is Galois, then so is L/K.
   (c) If K/F and L/K are both Galois, then so is L/F.

4. Let V be a finite dimensional vector space over a field (the field is not necessarily algebraically closed). Let $\varphi : V \to V$ be a linear transformation. Prove that there exists a decomposition of V as $V = U \oplus W$, where U and W are φ-invariant subspaces of V, $\varphi|_U$ is nilpotent, and $\varphi|_W$ is nonsingular.

5. Let A be an $n \times n$ matrix.
   (a) Suppose that v is a column vector such that the set $\{v, Av, \ldots, A^{n-1}v\}$ is linearly independent. Show that any matrix B that commutes with A is a polynomial in A.
   (b) Show that there exists a column vector v such that the set $\{v, Av, \ldots, A^{n-1}v\}$ is linearly independent if and only if the characteristic polynomial of A equals the minimal polynomial of A.

6. Let R be a commutative ring, and let M be an R-module. An R-submodule N of M is **maximal** if there is no R-module P with $N \subsetneq P \subsetneq M$.
   (a) Show that an R-submodule N of M is maximal iff M/N is a simple R-module: i.e., M/N is nonzero and has no proper, nonzero R-submodules.
   (b) Let M be a $\mathbb{Z}$-module. Show that a $\mathbb{Z}$-submodule N of M is maximal iff $\#M/N$ is a prime number.
   (c) Let M be the $\mathbb{Z}$-module of all roots of unity in $\mathbb{C}$ under multiplication. Show that there is no maximal $\mathbb{Z}$-submodule of M.

7. Let R be a commutative ring.

(a) Let $r \in R$. Show that the map $r\bullet : R \to R$ by $x \mapsto rx$ is an R-module endomorphism of R.

(b) We say that r is a **zero-divisor** if $r\bullet$ is not injective. Show that if r is a zero-divisor and $r \neq 0$, then the kernel and image of R each consist of zero-divisors.

(c) Let $n \geq 2$ be an integer. Show: if R has exactly n zero-divisors, then $\#R \leq n^2$.

(d) Show that up to isomorphism there are exactly two commutative rings R with precisely 2 zero-divisors. You may use without proof the following fact: every ring of order 4 is isomorphic to exactly one of the following: $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}[t]/(t^2+t+1)$, $\mathbb{Z}/2\mathbb{Z}[t]/(t^2-t)$, $\mathbb{Z}/2\mathbb{Z}[t]/(t^2)$.