(1) (a) (5 points) State the structure theorem for finitely generated modules over a principal ideal domain.
(b) (5 points) Find the decomposition of the $\mathbb{Z}$-module M generated by w, x, y, and z, and satisfying the relations

$$\begin{aligned} 3w + 12y + 3x + 6z &= 0 \\ 6y &= 0 \\ -3w - 3x + 6y &= 0. \end{aligned}$$

(2) (10 points) Let R be a commutative ring and let M be an R-module. Recall that for $\mu \in M$ the annihilator of μ is the set $Ann(\mu) = \{r \in R : r\mu = 0\}$. Suppose that I is an ideal in R which is maximal with respect to the property that there exists a nonzero element $\mu \in M$, such that $I = Ann(\mu)$. Prove that I is a prime ideal in R.

(3) (a) (5 points) Give the definition that a group G must satisfy to be solvable.
(b) (10 points) Show that every group G of order 36 is solvable. *Hint:* You may assume that S_4 is solvable.

(4) (15 points) Consider the matrix

$$A = \begin{pmatrix} 0&0&0&0&1 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{pmatrix}.$$

(a) Find the Jordan Normal Form of A regarded as a matrix over $\mathbf{C}$, the complex numbers.
(b) Find the Jordan Normal Form of A regarded as a matrix over $\mathbf{F}_5$, the field with five elements.

(5) (15 points) Let $F \subset L$ be fields such that L/F is a Galois field extension with Galois group equal to $D_8 =< \sigma, \tau : \sigma^4 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 >$. Show that there are fields $F \subset E \subset K \subset L$ such that E/F and K/E are Galois extensions, but K/F is not Galois.

(6) (15 Points) Let C/F be an algebraic field extension. Prove that the following are equivalent:
(a) Every nonconstant polynomial $f \in F[x]$ factors into linear factors in $C[x]$.
(b) For every (not necessarily finite) algebraic extension E/F there is a ring homomorphism $\alpha : E \to C$ that is the identity on F. *Hint:* Use Zorn's lemma.

(7) (10 Points) Let R be a commutative ring.
(a) Say what it means for R to be a unique factorization domain (UFD);
(b) Say what it means for R to be a principal ideal domain (PID);
(c) Give an example of a UFD that is not a PID. Prove that it is not a PID.

(8) (10 Points) Let p and q be distinct primes. Let k denote the smallest positive integer such that p divides $q^k - 1$. Prove that no group of order pq^k is simple.