Algebra qualifying exam, Spring 2007

Instructions: Work 8 problems.

1) a) If A is a 3×3 real matrix with $det A > 0$, prove A has a positive real eigenvalue.
b) Assuming a), if also $A^t A = Id$, prove A fixes some non zero vector v, ($Av = v$), and A defines a rotation about the line through v. [A rotation about a line L through the origin in $\mathbb{R}^3$ must fix L pointwise, preserve the perpendicular plane through the origin, and induce a rotation about the origin in this plane.]

2) If A is the 4×4 matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{bmatrix}$$

over $\mathbb{Q}$, then find the characteristic polynomial, minimal polynomial, Jordan form, and the cyclic decomposition for the $\mathbb{Q}[X]$ module structure defined by A on $\mathbb{Q}^4$. [The Jordan form of A does exist over $\mathbb{Q}$.]

3) a) Prove that an action by a group G on the set $\{1, ..., n\}$ defines a homomorphism $G \to Sym(n)$ from which the original action can be recovered.
b) Prove that if G has odd order, and H is a subgroup of index 3, then H is normal.

4) Prove in as much detail as possible, there exist exactly two groups of order 21 up to isomorphism, and describe both groups by generators and relations.

5) Prove directly that if k is a field, then $k[X]$ is a unique factorization domain, assuming basic divisibility properties for polynomials, but no general theorems about Euclidean domains, pid's and so on.

6) a) Prove the complex number field $\mathbb{C}$ contains exactly 5 subfields isomorphic to $\mathbb{Q}[X]/(X^5 - 3)$.
b) Compute the Galois group of $X^5 - 3$, and show it is a non abelian solvable group.

7) Prove the fundamental theorem of algebra as follows.
a) Assuming elementary calculus and high school algebra, explain why the real number field $\mathbb{R}$ has no non trivial odd degree extensions, and the complex number field $\mathbb{C}$ has no non trivial quadratic extensions.
b) Assuming a) prove that if E is any non trivial Galois extension of $\mathbb{R}$, then $E = \mathbb{C}$. [Hint: start by computing the fixed field of the Sylow 2-subgroup.]

8) a) If p and q are distinct primes, prove $\mathbb{Z}/(p^n q^m)$ and $\mathbb{Z}/(p^n) \times \mathbb{Z}/(q^m)$ are isomorphic as rings.
b) Find the standard decomposition of the abelian group $U = (\mathbb{Z}/(144))^*$ of units in the ring $\mathbb{Z}/(144)$, as a product of cyclic groups.

9) a) If f is a polynomial in $k[X]$, prove any two splitting fields E, F for f (both containing the field k), are isomorphic.
b) Prove two finite fields, both with exactly 32 elements, are isomorphic.