ALGEBRA QUALIFYING EXAM, JANUARY 2009

(1) Let G be a finite group, and $H < G$ a subgroup. Carefully show that the number of subgroups of G that are conjugate to H divides $|G|$.

(2) Let G be a group of order p^3 for some prime number p. Carefully show that either G is abelian or $|Z(G)| = p$.

(3) Let R be a commutative ring containing a field k and suppose that $\dim_k R < \infty$.

(a) Let $a \in R$. Show that there exist $n \in \mathbb{N}$ and $c_0, \ldots, c_{n-1} \in k$ such that $a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = 0$.

(b) Let $a \in R$. Suppose that there exist $n \in \mathbb{N}$ and $c_0, \ldots, c_{n-1} \in k$ with $a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = 0$. Show that if $c_0 \neq 0$, then a is a unit in R.

(c) Let $a \in R$. Suppose that there exist $n \in \mathbb{N}$ and $c_0, \ldots, c_{n-1} \in k$ with $a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = 0$. Show that if a is not a zero divisor in R, then a is invertible.

(4) Let R be a commutative domain.

(a) Define what it means for an element $r \in R$, $r \neq 0$, to be irreducible.

(b) Let P be a maximal ideal. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of positive degree in $R[x]$. Assume that $a_0, \ldots, a_{n-1} \in P$, and that $a_0 \notin P^2$. (Recall that P^2 denotes the ideal of R generated by elements of the form ab, with $a, b \in P$.) Show that $f(x)$ is irreducible in $R[x]$.

(c) (Extra credit only.) What happens if R is not assumed to be a domain?

(5) Let R be a commutative ring. Let M be an R-module.

(a) Define what a torsion element of M is.

(b) Give an example of a ring R with a cyclic R-module M with M infinite, and such that M contains a non-trivial torsion element m. (Justify why m is torsion.)

(c) Show that if R is a domain, then the subset of elements of M that are torsion is an R-submodule of M. Clearly show where the hypothesis that R is a domain is used.

(6) Let V denote the $\mathbb{R}$-vector space $\mathbb{R}[x]/((x-2)(x^2+3))$. The $\mathbb{R}$-vector space V can be considered in a natural way as an $\mathbb{R}[x]$-module.

(a) Let $L : V \to V$ denote the linear map defined as the 'multiplication-by-x map. Write down a basis in which L is in rational canonical form. Write down the matrix that represents L in that basis.

(b) Does L have a Jordan canonical form? If yes, find it, if not, explain why not.

(c) Let $T : V \to V$ denote the linear map defined as the 'multiplication-by-x^2 map. Write down a basis in which T is in rational canonical form. Write down the matrix that represents T in that basis.

(7) Let F be a field and let $f(x) \in F[x]$.

(a) Define what is a splitting field of $f(x)$ over F.

(b) Let F now be a finite field with q elements. Let E/F be a finite extension of degree $n > 0$. Exhibit an explicit polynomial $g(x) \in F[x]$ such that E/F is a splitting of $g(x)$ over F. Fully justify your answer.

(c) Show that the extension E/F in (b) is a Galois extension.

(8) Let $f(x) = x^3 - 7$ in each of the following parts:

(a) Let K be the splitting field for f over $\mathbb{Q}$. Describe the Galois group of $K/\mathbb{Q}$ and the intermediate fields between $\mathbb{Q}$ and K. Which intermediate fields are not Galois over $\mathbb{Q}$? Justify when needed.

(b) Let L be the splitting field for f over $\mathbb{R}$. What is the Galois group $L/\mathbb{R}$? Justify when needed.

(c) Let M be the splitting field for f over $\mathbb{F}_{13}$, the field with 13 elements. Describe the Galois group of $M/\mathbb{F}_{13}$. Justify when needed.