Algebra Qualifying Exam, Spring 2014

In this exam, all rings are assumed to have a multiplicative identity different from zero, and homomorphisms of rings are assumed to map the multiplicative identity to the multiplicative identity.

1. Let p and n be integers such that p is prime and p does not divide n. Find a real number $k = k(p, n)$ such that for every integer $m \geq k$, every group of order $p^m n$ is not simple.

2. Let $G \subset S_9$ be a Sylow 3-subgroup of the symmetric group on 9 letters.

i) Show that G contains a subgroup H isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ by exhibiting an appropriate set of cycles.

ii) Show that H is normal in G.

iii) Give generators and relations for G as an abstract group, i.e. $G = \langle x, y, ... \mid \text{relations} \rangle$, such that all your generators have order 3. Also exhibit elements of S_9 in cycle notation, corresponding to your abstract generators.

iv) Without appealing to the previous parts of the problem, prove that G contains an element of order 9.

3. Let $F \subset C$ be a field extension, with C algebraically closed.

i) Prove that the intermediate field $C_{alg} \subset C$ consisting of elements algebraic over F is algebraically closed.

ii) Prove that if $F \to E$ is an algebraic extension, there exists a homomorphism $E \to C$ that is the identity map on F.

4. Let $E \subset \mathbb{C}$ denote the splitting field over $\mathbb{Q}$ of the polynomial $x^3 - 11$.

i) Prove that if n is a square free positive integer, then $\sqrt{n} \notin E$. (Hint: You can describe all quadratic extensions of $\mathbb{Q}$ contained in E.)

ii) Find the Galois group of $(x^3 - 11)(x^2 - 2)$ over $\mathbb{Q}$.

iii) Prove that the minimal polynomial of $11^{1/3} + 2^{1/2}$ over $\mathbb{Q}$ has degree 6.

5. Let R be a commutative ring and let $a \in R$. Prove that a is not nilpotent if and only if there exists a commutative ring S and ring homomorphism $\phi : R \to S$ such that $\phi(a)$ is a unit. (Note: By

definition, a is nilpotent if and only if there is a natural number n such that $a^n = 0$.)

6. Let R be a commutative ring with identity and let n be a positive integer.

i) Prove that every surjective R-linear endomorphism of $R^n \xrightarrow{T} R^n$ is injective.

ii) Show that an injective R-linear endomorphism of R^n need not be surjective.

7. Let $G = GL_3(\mathbb{Q}[x])$ be the group of invertible 3×3 matrices over $\mathbb{Q}[x]$. For each $f \in \mathbb{Q}[x]$, let S_f be the set of 3×3 matrices A over $\mathbb{Q}[x]$ such that $det(A) = cf(x)$ for some nonzero constant $c \in \mathbb{Q}$.

i) Show that for $(P, Q) \in G \times G$ and $A \in S_f$, the formula

$$(P, Q) \cdot A = PAQ^{-1}$$

gives a well-defined map $G \times G \times S_f \to S_f$ and show that this map is an action of $G \times G$ on S_f.

ii) For $f(x) = x^3(x^2 + 1)^2$, give one representative from each orbit of the group action given in part i. Justify your assertion.