Complex Analysis Qualifying Exam - Spring 2018

Justify your answers and state clearly any theorem, proposition or lemma that you are applying. You should not cite examples, exercises, or problems from any source (other than this exam). Cross out the parts you do not want to be graded.

There are seven questions in this exam–answer them all.

Problem 1. Does there exist a conformal mapping of the region $\Omega_1 = \{z \in \mathbb{C} : \ |z-1| < 1/2\}$ onto the region $\Omega_2 = \{z \in \mathbb{C} : \ |z+1| > 1\}$?

Problem 2. Find all holomorphic functions: $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ such that that

$$|f(z)| < (|z|^{1/2} + |z|^{-1/2}) \quad \text{for all} \quad z \neq 0.$$

Problem 3. Suppose that $f(z)$ is an entire function and that $\lim_{z\to\infty} f(z) = \infty$. Prove that f must be a polynomial.

Problem 4. Show that

$$\int_0^\infty \frac{\log x}{1+x^4}\, dx = -\frac{\pi^2}{8\sqrt{2}}$$

Problem 5. Let $\mathbb{D}$ denote the unit open disk in the complex plane. Does there exist an analytic function $f : \mathbb{D} \to \mathbb{D}$ such that (i) $f(1/2) = 3/4$ and (ii) $f'(1/2) = 2/3$?

Problem 6 Let λ be a fixed complex number with $\mathrm{Re}(\lambda) > 1$. How many solutions the equation

$$e^z = z + \lambda$$

has in the left half-plane $\{z \in \mathbb{C} : \ \mathrm{Re}(z) < 0\}$?

Problem 7. Suppose that f is complex analytic in a punctured neighborhood U of z_0 except for poles at all points of a sequence $\{z_n\} \to z_0$. Show that $f(U)$ is *dense* in the complex plane (note: z_0 is <u>not</u> an isolated singularity of f).