Qualifying Exam: Complex Analysis — Spring 2015

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- Give clear reasoning. State clearly which theorems you are using.
- You should not cite anything else: examples, exercises, or problems.
- Cross out the parts you do not want to be graded.
- Read through all the problems, do them in any order, the one you feel most confident about first. They are not in order of difficulty. Each question weighs equally.

1. Prove that the function $f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$ provides a conformal equivalence from the open half disc $\mathbb{D}^+ := \{z = x + iy : \ |z| < 1, \ y > 0\}$ to the upper half plane $\mathbb{H} := \{z = x + iy : \ y > 0\}$.

2. Suppose n is an integer and a is a real number satisfying $0 < a < n$. Compute the integral $\int_0^\infty \frac{x^{a-1}}{1 + x^n}\, dx$.

3. Let $0 < r < 1$. Show that for all sufficiently large n, the polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$.

4. Let f be analytic in $\Omega : 0 < |z - a| < r$ except at a sequence of poles $a_n \in \Omega$ with $\lim_{n\to\infty} a_n = a$. Show that for any $w \in \mathbb{C}$, there exists a sequence $z_n \in \Omega$ such that $\lim_{n\to\infty} f(z_n) = w$.

5. Suppose f is entire and maps a circle $|z - a| = r$ into $\mathbb{R}$. Prove that f is constant.

6. Let $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z : \ |z| < 1\}$.
   Prove that $\frac{1}{\pi}\iint_{\mathbb{D}} |\psi'_\alpha| dxdy = \frac{1 - |\alpha|^2}{|\alpha|^2} \log \frac{1}{1 - |\alpha|^2}$.
   Hint: polar coordinates may come in handy.