Probability Theory, Ph.D Qualifying, Fall 2014

Completely solve any 5 problems.

1. (a) Let $\{X_n\}$ be a sequence of independent, identically distributed, nondegenerate random variables. Show that
$$P(\{X_n\} \text{ converges}) = 0.$$
(b) If random variables $\{X_n\}$ are uniformly integrable, show that so are $S_n/n = (\sum_{i=1}^{n} X_i)/n$.

2. If $\{A_n\}$ are events satisfying $P(A_n) = o(1)$, (here o(1) denotes a function tending to 0) and $\sum_{n=1}^{\infty} P(A_n A_{n+1}^c) < \infty$, show that
$$P(A_n, \text{ infinitely often}) = 0.$$

3. If $\{X_n\}$ a sequence of independent, identically distributed, $EX_1 = 0$, $E(|X_1| \log^+ |X_1|) < \infty$, then $\sum(X_n/n)$ converges a.s.

4. If the independent random variables $X_1, \ldots, X_n, \ldots$ satisfy the condition
$$V(X_i) \le c < \infty, \ i = 1, 2, \ldots,$$
then the SLLN (Strong Law of Large Numbers) holds.

5. Prove that for any random variable. X
$$E|X| = \int_0^\infty P(|X| \ge t)dt.$$

6. (a) State (without proof) the Levy continuity theorem regarding a sequence of characteristic functions.
(b) Let $\{X_n\}$ be independent, identically distributed random variables with distribution $F(x)$ having finite mean μ and variance σ^2. Let $S_n = X_1 + \cdots + X_n$. Show that
$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \to N(0,1) \text{ in distribution as } n \to \infty.$$

7. Let $\{X_n\}$ be independent, identically distributed random variables. Then,
(a) $n^{-1} \max_{1\le i\le n} |X_i| \to 0$ in probability if and only if $nP(|X_1| > n) = o(1)$.
(b) $n^{-1} \max_{1\le i\le n} |X_i| \to 0$ a.s. if and only if $E|X_1| < \infty$.