Numerical Analysis Qualifying Examination

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Each problem carries the same weight. Please start each problem on a separate page and label each problem on each page. The time limit on this examination is three hours.

Problem 1. Given $x_0 < x_1 < \cdots < x_n$, find the LU-factorization of the $(n+1) \times (n+1)$ Vandermonde matrix V with entries $V_{i,j} = x_j^i$, $i, j = 0, 1, \ldots, n$. It may be useful to exploit Newton's form for the polynomial $P(f)$ of degree $\leq n$ interpolating a function f at the points $x_0, x_1, \ldots, x_n$, namely

$$P(f)(x) = [x_0]f + [x_0, x_1]f \times (x - x_0) + \cdots + [x_0, x_1, \ldots, x_n]f \times (x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

Problem 2. Use Householder reflections or Givens rotations to determine the QR-factorization of the matrix

$$\mathbf{A} = \begin{bmatrix} 6 & 6 & 1 \\ 3 & 6 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Problem 3. Apply two iterations of Jacobi method to the system of linear equations $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 4 & 1 \\ 1 & 0 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Does the iterative process converge for the input data given above?

Problem 4. Let $\mathbf{A}$ be an $n \times n$ symmetric positive definite matrix. Given $\mathbf{b} \in \mathbb{R}^n$, the steepest descent algorithm for solving $\mathbf{Ax} = \mathbf{b}$ outputs the limit of a sequence $(\mathbf{x}_k)$ defined by

$$\mathbf{x}_k = \mathbf{x}_{k-1} + t_k \mathbf{v}_k \qquad \text{where } t_k \in \mathbb{R} \text{ and } \mathbf{v}_k \in \mathbb{R}^n \text{ with } \|\mathbf{v}_k\| = 1$$

are chosen so that the quadratic form $q(\mathbf{x}) := \langle \mathbf{x}, \mathbf{Ax} \rangle - 2\langle \mathbf{x}, \mathbf{b} \rangle$ decreases as much as possible at each iteration. Based on this rationale, derive expressions for the search direction $\mathbf{v}_k$ and for the step size t_k.

Problem 5. Given the three vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

find the representation of the vector $\mathbf{x} = [1, 0]^\top$ as a linear combination

$$\mathbf{x} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$$

providing the minimum value for $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$.

Problem 6. Under which conditions does Olver's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{f''(x_n) f^2(x_n)}{f'^3(x_n)}$$

provide cubic convergence when solving $f(x) = 0$? Establish the convergence under these conditions.

Problem 7. Find the precision estimate for the numerical differentiation formula

$$f'(x) \approx \frac{1}{12h}\big(8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h)\big).$$

Problem 8. Find the quadrature formula with nodes 0.25, 0.5, 0.75 that is exact on polynomials of maximum order for the integral

$$\int_0^1 f(x)\,dx.$$

What is this maximum order? What is the precision of this formula provided that the function f has an appropriate number of derivatives?

Problem 9. Given $f \in C^{n+1}[a,b]$ and $x_0 < x_1 < \cdots < x_n$ in $[a,b]$, establish the following expression for the approximation error of f by the polynomial p of degree $\leq n$ interpolating f at $x_0, x_1, \ldots, x_n$: for every $x \in [a,b]$, there exists $\xi \in [a,b]$ such that

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x - x_i).$$

Problem 10. For the differential equation $x' = f(t,x)$, derive the third order Runge–Kutta formula

$$x(t+h) = x(t) + \frac{1}{9}(2F_1 + 3F_2 + 4F_3),$$

where $F_1 = hf(t,x)$, $F_2 = hf(t + 0.5h, x + 0.5F_1)$, $F_3 = hf(t + 0.75h, x + 0.75F_2)$.