University of Georgia, Department of Mathematics

Fall, 2021

Numerical Analysis Qualifying Exam

Instruction: Please show all your work and put your name on all pages. All problems are 10 points each.

1. Let $f(x) = \sqrt{\pi x} - \cos(\pi x)$.
   (a) Show that the equation $f(x) = 0$ has at least one solution p in the interval $[0, 1]$.
   (b) Use Newton's method to solve $f(x) = 0$. Start the calculation with any chosen initial value p_0 and do three iterations to find p_3.
   (c) Justify the convergence order for the iterations.

2. Let $a = x_0 < x_1 < \cdots x_n < x_{n+1} = b$ be a partition of $[a, b]$. For $f \in C^1[a, b]$, let S_f be the C^1 cubic interpolatory spline of f, i.e.,

$$S_f(x_i) = f(x_i), S'_f(x_i) = f'(x_i),\ i = 0, 1, \cdots, n+1$$

   and $S_f(x)|_{[x_i, x_{i+1}]}$ is a cubic polynomial, $i = 0, \cdots, n$. Suppose that $f \in C^2[a, b]$. Show that

$$\int_a^b \left| \frac{d^2}{dx^2} (f(x) - S_f(x)) \right|^2 dx \leq \int_a^b \left| \frac{d^2}{dx^2} f(x) \right|^2 dx.$$

3. Derive the coefficients of a Gaussian quadrature formula of the form $\sum_{i=1}^n c_i f(x_i)$ with $n = 2$ to approximate the integral $\int_{-1}^1 f(x)dx$. Here c_i and x_i are all unknowns. What is the maximal degree of the polynomial for which this approximation is exact.

4. Find the least squares polynomial approximation of degree two over the interval $[-1, 1]$ for the function $f(x) = x^3 - 2x$. Note the first three Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

5. Let $f(0)$, $f(h)$, and $f(2h)$ be the values of a real valued function at $x = 0$, $x = h$, and $x = 2h$
   (a) Derive the coefficients c_0, c_1, and c_2 such that

$$Df_h(x) = c_0 f(0) + c_1 f(h) + c_2 f(2h)$$

   is as accurate an approximation to $f'(0)$ as possible.
   (b) Derive the leading term of a truncation error estimate for the above formula.

6. (a) Let A be an $n \times n$ matrix and $\| \cdot \|_1$ denote the standard ℓ_1 norm on $\mathbb{R}^n$ given by $\|v\|_1 = \sum_{i=1}^n |v_i|$. Show that

$$\|A\|_1 = \max_{j=1}^{n} \sum_{i=1}^{n} |A_{i,j}|.$$

(b) Let $\|\cdot\|_2$ denote the ℓ_2 norm on $\mathbb{R}^n$ given by $\|v\|_2 = (\sum_{i=1}^n |v_i|^2)^{1/2}$. For

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix},$$

compute $\|A\|_2$.

7. Find singular value decomposition (SVD) of the following matrix:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & i \\ 0 & i \end{pmatrix},$$

where i is an imaginary unit.

8. Consider the scalar second order equation for $u(x,t)$

$$au_{tt} + 2bu_{xt} + cu_{xx} = 0$$

to be solved for $t > 0$, $0 \le x < 1$ with periodic boundary conditions in x, i.e. $u(0,t) = u(1,t)$ for all $t > 0$ and initial data

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

with a, b and c given constants and $f(x)$ and $g(x)$ smooth.

(a) For what values of a, b and c is this problem well posed?

(b) Devise a convergence finite difference scheme to create approximate solutions to this problem.