University of Georgia, Department of Mathematics

Fall, 2021

Numerical Analysis Qualifying Exam

**Instruction:** Please show all your work and put your name on all pages. All problems are 10 points each.

1. Let $f(x) = \sqrt{\pi x} - \cos(\pi x)$.
   (a) Show that the equation $f(x) = 0$ has at least one solution $p$ in the interval $[0, 1]$.
   (b) Use Newton's method to solve $f(x) = 0$. Start the calculation with any chosen initial value $p_0$ and do three iterations to find $p_3$.
   (c) Justify the convergence order for the iterations.

2. Let $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$ be a partition of $[a, b]$. For $f \in C^1[a, b]$, let $S_f$ be the $C^1$ cubic interpolatory spline of $f$, i.e.,
   
   $$S_f(x_i) = f(x_i), \quad S_f'(x_i) = f'(x_i), \quad i = 0, 1, \cdots, n + 1$$

   and $S_f(x)_{|_{[x_i, x_{i+1}]}}$ is a cubic polynomial, $i = 0, \cdots, n$. Suppose that $f \in C^2[a, b]$. Show that
   
   $$\int_a^b \left| \frac{d^2}{dx^2} (f(x) - S_f(x)) \right|^2 dx \leq \int_a^b \left| \frac{d^2}{dx^2} f(x) \right|^2 dx.$$

3. Derive the coefficients of a Gaussian quadrature formula of the form $\sum_{i=1}^n c_i f(x_i)$ with $n = 2$ to approximate the integral $\int_{-1}^1 f(x)dx$. Here $c_i$ and $x_i$ are all unknowns. What is the maximal degree of the polynomial for which this approximation is exact.

4. Find the least squares polynomial approximation of degree two over the interval $[-1, 1]$ for the function $f(x) = x^3 - 2x$. Note the first three Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

5. Let $f(0)$, $f(h)$, and $f(2h)$ be the values of a real valued function at $x = 0$, $x = h$, and $x = 2h$
   (a) Derive the coefficients $c_0, c_1$, and $c_2$ such that
   
   $$Df_h(x) = c_0 f(0) + c_1 f(h) + c_2 f(2h)$$
   
   is as accurate an approximation to $f'(0)$ as possible.
   (b) Derive the leading term of a truncation error estimate for the above formula.

6. (a) Let $A$ be an $n \times n$ matrix and $\| \cdot \|_1$ denote the standard $\ell_1$ norm on $\mathbb{R}^n$ given by $\|v\|_1 = \sum_{i=1}^n |v_i|$. Show that
   
   $$\|A\|_1 = \max_{j=1}^n \sum_{i=1}^n |A_{i,j}|.$$
(b) Let \( \| \cdot \|_2 \) denote the \( \ell_2 \) norm on \( \mathbb{R}^n \) given by \( \| v \|_2 = (\sum_{i=1}^{n} |v_i|^2)^{1/2} \). For

\[
A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix},
\]

compute \( \|A\|_2 \).

7. Find singular value decomposition (SVD) of the following matrix:

\[
A = \begin{pmatrix} 1 & 0 \\ 1 & i \\ 0 & i \end{pmatrix},
\]

where \( i \) is an imaginary unit.

8. Consider the scalar second order equation for \( u(x,t) \)

\[
a u_{tt} + 2b u_{xt} + c u_{xx} = 0
\]

to be solved for \( t > 0, 0 \leq x < 1 \) with periodic boundary conditions in \( x \), i.e. \( u(0,t) = u(1,t) \) for all \( t > 0 \) and initial data

\[
u(x,0) = f(x), \quad u_t(x,0) = g(x)
\]

with \( a, b \) and \( c \) given constants and \( f(x) \) and \( g(x) \) smooth.

(a) For what values of \( a, b \) and \( c \) is this problem well posed?

(b) Devise a convergence finite difference scheme to create approximate solutions to this problem.