Mathematics Preliminary Exam, Fall 2016

1. A sequence of functions $\{f_n : \mathbb{R} \to \mathbb{R}\}_{n=1}^{\infty}$ *converges uniformly* to a function $f : \mathbb{R} \to \mathbb{R}$ if for all $\epsilon > 0$, there is a positive integer N such that for all $n > N$ and all $x \in \mathbb{R}$, we have $|f_n(x) - f(x)| < \epsilon$.
 a) What do you need to check to show that a sequence does not converge uniformly?
 b) Give an example such that for every $x \in \mathbb{R}$, $f_n(x)$ converges to $f(x)$, but the sequence does not converge uniformly.

2. Find an invertible matrix A and a diagonal matrix B such that $\begin{pmatrix} 4 & -6 \\ 3 & -5 \end{pmatrix} = ABA^{-1}$.

3. Give an example (with proof) of a power series that converges exactly on the interval $[3, 5)$.

4. Give an ϵ, δ proof that $\lim_{x \to 2} \dfrac{1}{3+x} = \dfrac{1}{5}$.

5. a) Choose a path C from $(2, 1)$ to $(3, 5)$ and compute $\int_C (2x + y)dx + ydy$.
 b) Give (and apply) a criterion that shows the above integral either is or is not path independent.

6. Use induction to show that for all positive integers $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$.

7. Let $f : X \to Y$ be a map of sets. Give, with proof, necessary and sufficient conditions so that for all subsets $S \subset X$, $f^{-1}(f(S)) = S$.

8. Draw representative level sets (i.e. $f^{-1}(c)$ for $c \in \mathbb{R}$) for $f(x, y) = xy$. Next, compute and draw the gradient, ∇f, at a representative collection of points in $\mathbb{R}^2$. Explain how these two objects must be related in general.

9. Give an example of a linear map defined on $\mathbb{R}^3$ with kernel generated by $(1, 3, 5)$.