Probability Theory, Ph.D Qualifying, Fall 2018

Completely solve any five problems.

1. Show that for any two random variables X and Y with $Var(X) < \infty$,

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y]).$$

2. Show that random variables $X_n,\ n \geq 1$, and X satisfy $X_n \to X$ in distribution iff

$$E[F(X_n)] \to E[F(X)]$$

for every continuous distribution function F.

3. If $\{A_n\}$ are events satisfying $\lim_{n\to\infty} P(A_n) = 0$ and $\sum_{n=1}^{\infty} P(A_n A_{n+1}^c) < \infty$, show that

$$P(A_n,\ \text{infinitely often}) = 0.$$

4. Prove for iid random variables $\{X_n\}$ with $S_n = X_1 + \cdots + X_n$ that

$$\frac{S_n - C_n}{n} \to 0 \text{ a.s.}$$

for some sequence of constants C_n if and only if $E[|X_1|] < \infty$.

5. Let $\{X_n\}$ be iid r.v.s with distribution $F(x)$ having finite mean μ and variance $\sigma^2 > 0$. Let $S_n = X_1 + \cdots + X_n$. Show that

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \to N(0,1) \text{ in distribution as } n \to \infty.$$

Here $N(0,1)$ is a standard normal random variable.

6. Suppose that $\{X_n,\ n \geq 1\}$ is a sequence of independent identically distributed random variables with $E[X_1] = 0$. Prove that

$$P\left(\frac{X_n}{n^{1/\alpha}} \to 0 \text{ as } n \to \infty\right) = 1, \alpha > 0,$$

if and only if $E[|X_1|^\alpha] < \infty$.

7. Let $\{\xi_n,\ n \geq 1\}$ be independent random variables such that for some $0 < p < 1$, $P(\xi_n = 1) = p$, $P(\xi_n = -1) = 1 - p = q$. For $n \geq 1$, let $\eta_n = \sum_{k=1}^{n} \xi_k$ and $\zeta_n = (q/p)^{\eta_n}$. Show that $\{\zeta_n,\ n \geq 1\}$ is a martingale.