1. Prove that if \( f : [0, 1] \to \mathbb{R} \) be continuous, then
\[
\lim_{k \to \infty} \int_0^1 kx^{k-1} f(x) \, dx = f(1).
\]

2. Let \( m_* \) denote Lebesgue outer measure on \( \mathbb{R} \).
   (a) Prove that for every \( E \subseteq \mathbb{R} \) there exists a Borel set \( B \) containing \( E \) with the property that
   \[
   m_*(B) = m_*(E).
   \]
   (b) Prove that if \( E \subseteq \mathbb{R} \) has the property that
   \[
   m_*(A) = m_*(A \cap E) + m_*(A \cap E^c)
   \]
   for every set \( A \subseteq \mathbb{R} \), then there exists a Borel set \( B \subseteq \mathbb{R} \) such that \( E = B \setminus N \) with \( m_*(N) = 0 \).
   Make sure you address the case when \( m_*(E) = \infty \).

3. (a) Prove that if \( f \in L^1(\mathbb{R}) \), then
   \[
   \lim_{N \to \infty} \int_{|x| \geq N} |f(x)| \, dx = 0,
   \]
   and demonstrate that it is not necessarily the case that \( f(x) \to 0 \) as \(|x| \to \infty\).
   (b) Prove that if \( f \in L^1([1, \infty)) \) and decreasing, then \( \lim_{x \to \infty} f(x) = 0 \) and in fact \( \lim_{x \to \infty} xf(x) = 0 \).
   (c) If \( f : [1, \infty) \to [0, \infty) \) is decreasing with \( \lim_{x \to \infty} xf(x) = 0 \), does this ensure \( f \in L^1([1, \infty)) \)?

4. Let \( f \in L^1(\mathbb{R}) \) and \( g \in L^1(\mathbb{R}) \). Argue that \( H(x, y) = f(y)g(x - y) \) defines a function in \( L^1(\mathbb{R}^2) \) and deduce from this that
   \[
   f \ast g(x) = \int_{\mathbb{R}} f(y)g(x - y) \, dy
   \]
   defines a function in \( L^1(\mathbb{R}) \) that satisfies
   \[
   \|f \ast g\|_1 \leq \|f\|_1 \|g\|_1.
   \]

5. Compute the following limit and justify your calculations:
   \[
   \lim_{n \to \infty} \int_0^n \left( 1 + \frac{x^2}{n} \right)^{-(n+1)} \, dx.
   \]

6. (a) Show that \( L^2([0, 1]) \subseteq L^1([0, 1]) \) and \( \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}) \).
   (b) For \( f \in L^1([0, 1]) \) define
   \[
   \hat{f}(n) := \int_0^1 f(x)e^{-2\pi inx} \, dx.
   \]
   Prove that if \( f \in L^1([0, 1]) \) and \( \{\hat{f}(n)\} \in \ell^1(\mathbb{Z}) \), then
   \[
   S_N f(x) = \sum_{|n| \leq N} \hat{f}(n) e^{2\pi inx}
   \]
   converges uniformly on \([0, 1]\) to a continuous function \( g \) that equals \( f \) almost everywhere.
   \textit{Hint: One possible approach is to argue that if \( f \in L^1([0, 1]) \) with \( \{\hat{f}(n)\} \in \ell^1(\mathbb{Z}) \), then \( f \in L^2([0, 1]) \).}