PRINT NAME: ______________________________

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Analysis Qualifying Exam: Real Analysis

Give clear reasoning. State clearly which theorem you are using. Cross out the parts you do not want to be graded. Read through all the problems, do them in any order, the one you feel most confident about first. They are not in the order of difficulty. You should not cite anything else: examples, exercises, or problems.

Problem #	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Committee Recommendation

Grader's Remark

1. Let $f(x)$ denote the series $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$:

(a) Prove that this series does **not** converge uniformly on $[0, +\infty)$;

(b) Prove that $f(x)$ is continuous on $[0, +\infty)$.

2. Suppose that $E \subset \mathbb{R}^d$ is a measurable subset in $\mathbb{R}^d$ and $m(E) < \infty$ and $E = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$. Assume that $m(E) = m_*(E_1) + m_*(E_2)$. Here m_* is the exterior Lebesgue measure on $\mathbb{R}^d$. Prove that E_1 and E_2 are measurable.

3. Let f be a locally integrable function $\mathbb{R}^d$. Then the *maximal function* is defined to be

$$f^*(x) = \sup_{B \ni x} \frac{1}{m(B)} \int_B |f(y)|dy$$

where the supremum is taken over all balls containing the point x.

(a) Suppose that $f \in L^1(\mathbb{R}^d)$ and f is not identically zero. Show that $f^*(x) \geq c/|x|^d$ for some $c > 0$ and all $|x| \geq 1$.

(b) Let $\{K_\delta\}_{\delta>0}$ be a family of integrable functions in $\mathbb{R}^d$ with the following properties:

(i) $|K_\delta| \leq A\delta^{-d}$ for all $\delta > 0$.

(ii) $|K_\delta| \leq A\delta/|x|^{d+1}$ for all $\delta > 0$ and $x \in \mathbb{R}^d$.

Prove that

$$\sup_{\delta>0} |(f * K_\delta)(x)| \leq cf^*(x)$$

for some constant $c > 0$ and all integrable f. Here the convolution $f * K_\delta(x)$ is given by $f * K_\delta(x) = \int_{\mathbb{R}^d} f(x-y)K_\delta(y)dy$.

4. Let $\{\phi_i\}$ be a countable subset of an infinite dimensional Hilbert space H. Does Parseval's identity for $\{\phi_i\}$:

$$(f, f) = \sum_{i=1}^{\infty} |(f, \phi_i)|^2 \quad \text{for all } f \in H$$

imply that $\{\phi_i\}$ constitutes an orthonormal basis? Prove or give a counterexample.

5. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measurable space, and $p \leq 1 < \infty$. The space $L^p(X, \mathcal{M}, \mu)$ (denoted by L^p) consists of all complex-valued measurable functions on X that satisfy $\int_X |f(x)|^p d\mu(x) < \infty$. We define the L^p norm of f by $\|f\|_p = \left(\int_X |f(x)|^p d\mu(x) < \infty\right)^{1/p}$.

(a) Let A and B be two non-negative real numbers, and $0 \leq \theta \leq 1$, Prove that $A^\theta B^{1-\theta} \leq \theta A + (1-\theta)B$.

(b) (Hölder's inequality) Let $1 < p < \infty$ and $1 < q < \infty$ be conjugate exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p$ and $g \in L^q$. Prove that $fg \in L^1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

(c) (Minkowski's inequality). If $1 \leq p < \infty$ and $f, g \in L^p$. Prove that $f + g \in L^p$ and $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. (Hint: $|f + g|^p \leq (|f| + |g|)|f + g|^{p-1}$).