1. Prove that
\[ f(x) = \sum_{n=1}^{\infty} \frac{x}{1 + n^2 x^2} \]
defines a function that is differentiable with a continuous derivative on \((0, \infty)\) and that
\[ f'(x) = \sum_{n=1}^{\infty} \frac{1 - n^2 x^2}{(1 + n^2 x^2)^2} \]
on \((0, \infty)\).

2. (a) Let \(E\) be a subset of \(\mathbb{R}^d\) with the property that \(E \cap \{x \in \mathbb{R}^d : |x| \leq k\}\) is closed for all \(k \in \mathbb{N}\). Prove that \(E\) must itself be closed.

(b) Let \(\mu\) be a Borel measure on \(\mathbb{R}^d\) that assigns finite measure to all bounded Borel sets. Prove that for every \(F_\sigma\) set \(V \subseteq \mathbb{R}^d\) and \(\varepsilon > 0\) there exists a closed set \(F \subseteq V\) such that \(\mu(V \setminus F) < \varepsilon\).

3. Let \(f \in L^1(\mathbb{R})\). Prove that
\[ \lim_{n \to \infty} \int_{\mathbb{R}} |f(x)|^{1/n} \, dx = m(\{x \in \mathbb{R} : f(x) \neq 0\}) \]
where we are allowing for the possibility that both sides equal infinity.

4. Let \(f, g \in L^2([0, 1])\). Prove that if
\[ \int_0^1 f(x) x^n \, dx = \int_0^1 g(x) x^n \, dx \]
for all integers \(n \geq 0\), then \(f = g\) almost everywhere.

5. Prove that if \(f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\) and \(g \in L^1(\mathbb{R})\), then
\[ f \ast g(x) := \int_{\mathbb{R}} f(x-y)g(y) \, dy \]
defines a function in \(L^1(\mathbb{R}) \cap L^2(\mathbb{R})\) that satisfies the following estimates:
(a) \( \|f \ast g\|_1 \leq \|g\|_1 \|f\|_1 \)
(b) \( \|f \ast g\|_2 \leq \|g\|_1 \|f\|_2 \)

Hint: For the second estimate first argue that \( |f \ast g|^2 \leq \|g\|_1 (|f|^2 * |g|) \)

6. (a) Prove that if \(E \subseteq \mathbb{R}\) with \(m(E) > 0\), then
\[ \int_E e^{-\pi x^2} \, dx > 0. \]

(b) Let \(f \in L^\infty(\mathbb{R})\). Prove that
\[ \lim_{p \to \infty} \left( \int_{\mathbb{R}} |f(x)|^p e^{-\pi x^2} \, dx \right)^{1/p} = \|f\|_\infty. \]