Epistolary Math Tournament - Fall MMXXI

University of Georgia

Friday December 3th

Set 3 - Solution

Problem 1

Recall how the long division algorithm works:

$$
\begin{array}{r}
0673\\
3\overline{)2021}\\
\underline{0}\\
{\color{blue}2}0\\
\underline{18}\\
{\color{blue}2}2\\
\underline{21}\\
{\color{blue}1}1\\
\underline{09}\\
{\color{blue}2}
\end{array}
$$

The blue numbers are the remainders after each step of the long division. We say that this division generates the sequence of remainders $2, 2, 1, 2$. Notice the unusual leading 0 in the quotient. This guarantees that the number of remainders equals the number of digits.

(a) (1pt) What is the sequence of remainders generated by dividing 5624 by 3?

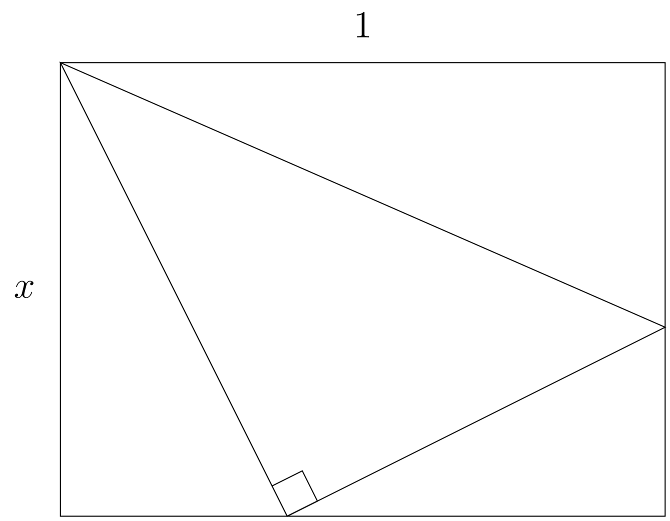
(b) (4pts) When the positive integer n is divided by 11, the sequence of remainders generated is $2, 5, 7, 1, 1, 7$. What is n?

(c) (5pts) When the positive integer m is divided by 3, the sequences of remainders is $0, 2, 0, 0$. When m is divided by 7 the sequence of remainders is $2, 0, 0, 6$. What is m?

Problem 2

(a) (3pts) Suppose that the three sides of a right triangle are in geometric progression: a, ar, ar^2. What is the sine of the smallest angle?

(b) (7pts)



In the above figure, the rectangle is partitioned into four similar triangles. What is the value of x? The diagram is not drawn to scale.

Problem 3

(a) (2pts) Write the current year, 2021, in binary (base 2).

(b) (8pts) Recall that the golden ratio is the largest root of $p(x) = x^2 - x - 1$. Its exact value is $\frac{1+\sqrt{5}}{2}$, and its decimal value is approximately 1.618. What are the first 9 digits of its binary expansion? The first (leftmost) digit is clearly 1 (why?), so your answer will be in the form $1.a_1a_2\ldots a_8$.

Solution 1

(a) Perform the long division, marking in blue the remainders as they arise:

$$\begin{array}{l}
\;1874 \\
3\overline{)5624} \\
\;\underline{3} \\
\;26 \\
\;\underline{24} \\
\;22 \\
\;\underline{21} \\
\;14 \\
\;\underline{12} \\
\;2
\end{array}$$

So the sequence of remainders is $2, 2, 1, 2$. Notice that this is the same sequence that appeared in the example which shows that the sequence of remainders does **not** determine the dividend.

(b) Since the sequence of remainders has 6 terms, this says that n is a 6 digit number, and its first (leftmost) digit is 2. (As in our example, the leftmost digit of the quotient is 0). Writing $n = a_1 a_2 \dots a_6 = 2a_2 \dots a_6$ where $0 \le a_i \le 9$, the sequence of remainders tells us:

$$a_1 \equiv 2 \pmod{11}$$

$$a_1 a_2 \equiv 5 \pmod{11}$$

$$a_1 a_2 a_3 \equiv 7 \pmod{11}$$

$$a_1 a_2 a_3 a_4 \equiv 1 \pmod{11}$$

$$a_1 a_2 a_3 a_4 a_5 \equiv 1 \pmod{11}$$

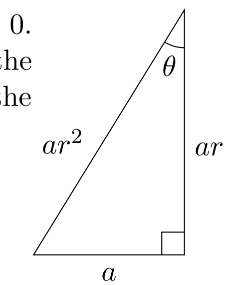
$$a_1 a_2 a_3 a_4 a_5 a_6 \equiv 7 \pmod{11}$$

Beware that $a_1 a_2$ is **not** a product! Starting from a_1, these congruences tell us that $a_1 = 2$, $a_2 = 7$, $a_3 = 1$, $a_4 = 8$, $a_5 = 2$ and $a_6 = 8$.

So $n = 271828$.

Solution 2

(a) Take the lengths of the sides of the triangle to be a, ar, ar^2, with $a \neq 0$. We can assume, without loss of generality, that $r > 1$. Then a is the shortest side, ar^2 is the hypotenuse and we're looking for the sine of the angle θ opposite side a. This is given by



$$\sin\theta = \frac{a}{ar^2} = \frac{1}{r^2}.$$

Note that this is independent of a. Using the Pythagorean Theorem to determine r, we have

$$a^2 + (ar)^2 = (ar^2)^2$$
$$\Rightarrow 1 + r^2 = r^4.$$

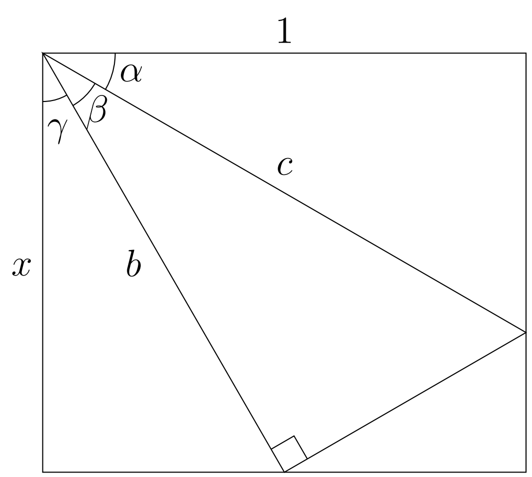
Let $s = r^2$, and this becomes

$$1 + s = s^2.$$

The quadratic formula gives $s = \frac{1\pm\sqrt{5}}{2}$; since $r > 1$ by assumption and so $s > 1$ we must have that $s = r^2 = \frac{1+\sqrt{5}}{2}$. Thus,

$$\sin\theta = \frac{1}{r^2} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}.$$

(b) The figure below is drawn to scale.



Notice that the three angles α, β, γ in the upper-left corner are congruent; if two of them were unequal, say, $\alpha \neq \beta$, then they would be complementary, so $\alpha+\beta = 90°$, leaving no room for γ. Since $\alpha = \beta = \gamma$ and $\alpha + \beta + \gamma = 90°$, then $\alpha = \beta = \gamma = 30°$. It follows that all triangles are 30-60-90 triangles. By triangle similarity we then have

$$x = \frac{x}{1} = \frac{b}{c} = \cos(30°) = \frac{\sqrt{3}}{2}.$$

Solution 3

(a) Begin with the powers of two:

$$2^0 = 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024 = 2^{10}.$$

Notice that $2^{11} > 2021$ so the coefficients in front of 2^n will be 0 when $n > 10$. We can now apply the greedy algorithm to write 2021 as a sum of powers of 2: $2021 - 1024 = 997$, $997 - 512 = 485$, $485 - 256 = 229$, $229 - 128 = 101$, $101 - 64 = 37$, $37 - 32 = 5$, $5 - 4 = 1$. In short, $2021 = 1024 + 512 + 256 + 128 + 64 + 32 + 4 + 1$, or $2021 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^2 + 2^0$:

$$2021 = (11111100101)_2$$

.

Alternatively, we can notice that $2021 = 2048 - 1 - 26$. In binary, $2048 - 1 = 2^{11} - 1 = (11111111111)_2$ and $26 = 2^4 + 2^2 + 2 = (11010)_2$. So, $2021 = (11111111111)_2 - (11010)_2 = (11111100101)_2$

(b) Let $\gamma = \frac{1+\sqrt{5}}{2}$ be the golden ratio. So, $\gamma^2 - \gamma - 1 = 0$ or, equivalently $\gamma(\gamma - 1) = 1$. We'll use this property to determine the first nine binary digits $a_0.a_1 \dots a_8$ of γ. From $\gamma \approx 1.618$ we know that $a_0 = 1$ and we need

$$(1.a_1 \dots a_8 \dots)(0.a_1 \dots a_8 \dots) = 1.$$

Notice first that if $a_1 = 0$ then,

$$1 = (1.0a_2 \dots)(.0a_2 \dots) = (.0a_2 \dots) + (.0a_2 \dots)^2 \leq (0.1) + (0.1)^2 = 0.11$$

which is a contradiction. So $a_1 = 1$. Similarly, if $a_2 = 1$, we get another contradiction:

$$1 = (1.11a_3 \dots)(0.11a_3 \dots) \geq (1.11)(0.11) = 1.010$$

So $a_2 = 0$. In the same vein, if $a_3 = 1$ we obtain a contradiction, etc. Continuing in this fashion, you'll find

$$\gamma = 1.10011110 \dots$$