Topology Qualifying Exam

January 2015

Instructions: Work all eight problems; justify your calculations and state the theorems you use. The problems are weighted equally.

(1) Prove that the product of two connected topological spaces is connected.

(2) Let X be the topological space obtained as the quotient space of a regular $2n$–gon ($n \geq 2$) in $\mathbb{R}^2$ by identifying opposite edges via translations in the plane. First show that X is a compact, orientable surface without boundary and then identify its genus as a function of n.

(3) Let S^1 denote the unit circle in $\mathbb{C}$, X be any topological space, $x_0 \in X$, and $\gamma_0, \gamma_1 : S^1 \to X$ two continuous maps such that $\gamma_0(1) = \gamma_1(1) = x_0$. Prove that γ_0 is homotopic to γ_1 if and only if the elements represented by γ_0 and γ_1 in $\pi_1(X, x_0)$ are conjugate.

(4)
 (a) Prove that a topological space which has a countable base for its topology also contains a countable dense subset.
 (b) Prove that the converse to (a) holds if the space is a metric space.

(5) Let X be the topological space constructed by attaching a closed 2-disk D^2 to the circle S^1 by a continuous map $\partial D^2 \to S^1$ of degree $d > 0$ on the boundary circle.
 (a) Show that every continuous map $X \to X$ has a fixed point.
 (b) Explain how to obtain all the connected covering spaces of X.

(6) Let X denote the quotient space formed from the sphere S^2 by identifying two distinct points. Compute the fundamental group and the homology groups of X.

(7) Define a family $\mathcal{T}$ of subsets of $\mathbb{R}$ by saying that $A \in \mathcal{T}$ if and only if either $A = \varnothing$ or $\mathbb{R} \setminus A$ is a finite set. Prove that $\mathcal{T}$ is a topology on $\mathbb{R}$, and that $\mathbb{R}$ is compact with respect to this topology.

(8) Let X be a topological space and let $U, V \subset X$ be open subsets with $X = U \cup V$. Prove that the Euler characteristics of U, V, $U \cap V$, and X obey the relation

$$\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V)$$

(You may assume that the homologies of $U, V, U \cap V, X$ are finite-dimensional so that their Euler characteristics are well-defined.)