(1) Prove that a compact subset of a Hausdorff space is closed, and give a counterexample showing that the Hausdorff assumption is necessary.

(2) Recall that the one-point compactification $\bar{X}$ of a topological space $X$ is given by the following.

- As a set $\bar{X} = X \cup \{\infty\}$, where $\infty$ is a point.
- A subset is open in $\bar{X}$ if it is either an open set of $X$, or is of the form $U \cup \{\infty\}$ where $U \subset X$ has the property that $X - U$ is compact.

Prove that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to the sphere $S^n$.

(3) Consider two 2-spheres $\Sigma_1$ and $\Sigma_2$ with their respective north and south poles $N_1, N_2$ and $S_1, S_2$. Take the quotient space $A := \Sigma_1 \sqcup \Sigma_2 / \sim$, where $N_1 \sim N_2$ and $S_1 \sim S_2$. Describe all covering spaces of $A$.

(4) Show that the double covering is a double cover of the Klein bottle. Show also that the double covering map is not homotopic to the constant map.

(5) Show that there is no continuous map $f : S^2 \rightarrow S^2$ such that $f \circ f$ has degree $-1$.

- Find a continuous map $f : T^2 \rightarrow T^2$ so that the induced map $(f \circ f)_* : H_1(T^2) \rightarrow H_1(T^2)$ is given by multiplication by $-1$.

(6) Let $A \subset X$ be a pair of topological spaces. Recall that a map $r : X \rightarrow A$ is called a retraction if $r(a) = a$ for all $a \in A$. Assume such a retraction exists below.

- If $X = D^2$, show that $H_i(A) = 0$ for $i > 0$ and $H_0(A) = \mathbb{Z}$.
- If $X$ is an annulus, list all possible $H_i(A)$ for $i \geq 0$. Prove that your list is exhaustive.

(7) For a topological space $X$, the suspension $S(X)$ is defined as $([0,1] \times X)/\sim$, where $(c, x) \sim (c, x)$ for either $c = 0$ or $c = 1$.

- Prove that $S(S^n)$ is homeomorphic to $S^{n+1}$.
- Find $H_1(S(T^2))$. 

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(8) For a map $f : X \to Y$, define the **graph of** $f$ as

$$G(f) := \{ (x, y) \in X \times Y : y = f(x) \}.$$ 

Let $f : S^1 \to S^1$ be a continuous map with degree $d$. Find, in terms of $d$, the relative homology $H_*(S^1 \times S^1, G(f))$. 