Topography Qualification Exam, Fall 2021

Instructions:

(a) Please work on 8 out of 9 problems, and clearly mark which one you do not want us to grade.

(b) You can assume homology groups and fundamental groups of a point and wedges of spheres in all dimensions. Everything else should be computed.

1. Prove or disprove the following:
   
   (a) If $X$ and $Y$ are path-connected, then $X \times Y$ is path-connected.
   
   (b) If $A \subset X$ is path-connected, then its closure $\overline{A}$ is path-connected.

2. Let $S$ be a connected metric space with metric $d$. Given $p \in S$, show that if $S \setminus \{p\} \neq \emptyset$ then $S \setminus \{p\}$ is not compact.

3. Given an example of a continuous map $f : X \to Y$ between connected spaces that is a continuous bijection but not a homeomorphism.

4. (a) Compute fundamental groups of $T^3$ and $\mathbb{R}P^3$ (Hint: construct their universal covers.).
   
   (b) Prove there is no covering map from $T^3$ to $\mathbb{R}P^3$.

5. Let $\Sigma_g$ denote the surface of genus $g$.
   
   (a) Suppose there is a degree $n$ covering map $f : \Sigma_g \to \Sigma_h$. What is the relationship between $g, h$ and $n$?
   
   (b) Show that there is no finite covering map from $\Sigma_{g+1}$ to $\Sigma_g$ for $g > 2$.

6. Let $X$ be the topological space obtained from the Klein bottle $K$ by removing a small open disk and identifying antipodal points of the resulting boundary circle on $K$ as in the following figure.

   (a) Use Van Kampen’s theorem to find a presentation for $\pi_1(X)$.
   
   (b) Compute the homology groups using cellular homology.

7. Let $X$ be the topological space obtained by gluing the boundary of a disk to a torus along a figure eight shape curve as in the following figure. Use the Mayer-Vietoris sequence to compute the homology groups of $X$. 
8. (a) Compute the homology groups of $X = S^2 \times S^4$ and $Y = \mathbb{C}P^2 \vee S^6$.

(b) Show that $X$ and $Y$ are not homeomorphic.

9. Consider the torus $T$ in $\mathbb{R}^3$ obtained by revolving the circle $(y - 2)^2 + z^2 = 1$ in the $yz$-plane around the $z$-axis. Let $i$ be the map induced by 180°-rotation around the $y$-axis on this torus i.e.,

$$i(x, y, z) = (-x, y, -z)$$

(a) Find a cell structure on $T$ such that $i$ maps cells to cells.

(b) The quotient of $T$ with the relation $x \sim i(x)$ for all $x \in T$ is an orientable surface (you do not need to show this, you can take this as given). Find the genus of this surface.