1. Let $X$ be a topological space, give $X \times X$ the product topology, and let
\[ \Delta := \{(x, x) \in X \times X \mid x \in X\}. \]
Show that the following are equivalent:
(i) The set $\Delta$ is closed in $X \times X$.
(ii) The space $X$ is Hausdorff.

2. Let $X = \prod_{n=1}^{\infty} \{0, 1\}$, endowed with the product topology.
(a) Show that for all points $x, y \in X$ with $x \neq y$, there are open subsets $U_x, U_y$ of $X$ such that $x \in U_x, y \in U_y, U_x \cup U_y = X$, and $U_x \cap U_y = \emptyset$.
(b) Show that $X$ is totally disconnected: the only nonempty connected subsets of $X$ are $\{x\}$ for $x \in X$.

3. For nonempty subsets $A$ and $B$ of a metric space $(X,d)$ the setwise distance is
\[ d(A, B) := \inf\{d(a, b)\mid a \in A, b \in B\}. \]
(a) Suppose that $A$ and $B$ are compact. Show that there is $a \in A$ and $b \in B$ such that $d(a, b) = d(A, B)$.
(b) Suppose that $A$ is closed and $B$ is compact. Show: if $d(A, B) = 0$ then $A \cap B \neq \emptyset$.
(c) Give an example in which $A$ is closed, $B$ is compact, and $d(a, b) > d(A, B)$ for all $a \in A$ and $b \in B$. (Suggestion: take $X = \{0\} \cup (1, 2] \subset \mathbb{R}$.)

4. Suppose that $X$ is a topological space, that $x_0 \in X$, and that every continuous map $\gamma : S^1 \to X$ is freely homotopic to the constant map to $x_0$. Prove that $\pi_1(X, x_0) = \{e\}$.

5. Identify five mutually non-homeomorphic connected spaces $X$ for which there is a covering map $p : X \to K$, where $K$ is the Klein bottle, giving an example of a corresponding covering in each case.

6. For each of the following spaces, compute the fundamental group and the homology groups.
(a) The graph $\Theta$ consisting of two vertices and three edges connecting these vertices.
(b) The two-dimensional cell complex $\Theta_2$ consisting of a closed circle and three two-dimensional disks each having boundary running once around that circle.

7. Prove directly from the definition that the zeroth singular homology of a nonempty path-connected space is isomorphic to $\mathbb{Z}$.

8. Let $\Sigma_{g,n}$ denote the compact oriented surface of genus $g$ with $n$ boundary components.
(a) Show that $\Sigma_{0,3}$ and $\Sigma_{1,1}$ are both homotopy equivalent to $S^1 \vee S^1$.
(b) Give a complete classification of pairs $(g, n)$ and $(g', n')$ for which $\Sigma_{g,n}$ is homotopy equivalent to $\Sigma_{g',n'}$.

9. Prove that for every continuous map $f : S^{2n} \to S^{2n}$ there is $x \in S^{2n}$ so that either $f(x) = x$ or $f(x) = -x$. You may use standard facts about degrees of maps of the sphere, including that the antipodal map of $S^{2n}$ has degree $-1$.

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1 Throughout this problem you may use without proof the continuity of $d : X \times X \to \mathbb{R}$.
2 i.e., with no conditions on basepoints.