ALGEBRA QUALIFYING EXAM, SPRING 2013

Instructions: Complete all 8 problems. In multi-part problems, you may assume the result of any part (even if you have not been able to do it) in working on subsequent parts.

(1) Let R be a commutative ring.
  (a) Define a maximal ideal, and prove that R has a maximal ideal.
  (b) Show that an element $r \in R$ is not invertible if and only if it is contained in a maximal ideal.
  (c) Let M be an R-module. Recall that for $\mu \in M$, $\mu \neq 0$, the annihilator of μ is the set $Ann(\mu) = \{r \in R : r\mu = 0\}$. Suppose that I is an ideal in R which is maximal with respect to the property that there exists an element $\mu \in M$, such that $I = Ann(\mu)$, for some $\mu \in M$. (In other words, $I = Ann(\mu)$ but there does not exist $\nu \in M$ with $J = Ann(\nu) \subsetneq R$ such that $I \subsetneq J$.) Prove that I is a prime ideal in R

(2) (a) Define a Euclidean Domain.
  (b) Define a Unique Factorization Domain.
  (c) Is a Euclidean Domain also a Unique Factorization Domain? Give either a proof or a counter example (with justification).
  (d) Is a Unique Factorization Domain also a Euclidean Domain? Give either a proof or a counter example (with justification).

(3) Let P be a finite p-group. Prove that every nontrivial normal subgroup of P intersects the center of P nontrivially.

(4) Define simple group. Prove that a group of order 56 can not be simple.

(5) Let $T : V \to V$ be a linear map from a 5-dimensional complex vector space to itself, and suppose that $f(T) = 0$ where f is the polynomial $f(x) = x^2 + 2x + 1$.
  (a) Show that there does not exist any vector $v \in V$ such that $Tv = v$, but that there does exist a vector $w \in V$ such that $T^2 w = w$.
  (b) Give all the possible Jordan canonical forms of a linear transformation T which satisfies the above relation $f(T) = 0$.

(6) Let V be a finite dimensional vector space over the field F and let $T : V \to V$ be a linear operator with characteristic polynomial $f(x) \in F[x]$.
  (a) Show that $f(x)$ is irreducible in $F[x] \Leftrightarrow$ there are no proper nonzero subspaces W of V with $T(W) \subseteq W$.
  (b) If $f(x)$ is irreducible in $F[x]$ and the characteristic of F is 0, show that T is diagonalizable when we extend the field to its algebraic closure.

(7) Let $f(x) = g(x)h(x) \in \mathbb{Q}[x]$. Let $E/\mathbb{Q}$, $B/\mathbb{Q}$, and $C/\mathbb{Q}$ be splitting fields of $f(x)$, $g(x)$ and $h(x)$, respectively.
  (a) Prove that $\mathrm{Gal}(E/B)$ and $\mathrm{Gal}(E/C)$ are normal subgroups of $\mathrm{Gal}(E/\mathbb{Q})$.
  (b) Prove that $\mathrm{Gal}(E/B) \cap \mathrm{Gal}(E/C) = \{1\}$.

(c) If $B \cap C = \mathbb{Q}$ show that $\mathrm{Gal}(E/B)\mathrm{Gal}(E/C) = \mathrm{Gal}(E/\mathbb{Q})$.
(d) Under the hypothesis of (c) show that $\mathrm{Gal}(E/\mathbb{Q}) \cong \mathrm{Gal}(E/B) \times \mathrm{Gal}(E/C)$.
(e) Use (d) to describe $\mathrm{Gal}(\mathbb{Q}[\alpha]/\mathbb{Q})$ where $\alpha = \sqrt{2} + \sqrt{3}$.

(8) Let F be the field of 2 elements and K a splitting field of $f(x) = x^6 + x^3 + 1$ over F. This polynomial is known to be irreducible (you may assume this).
(a) Show that if r is a root of f in K, then $r^9 = 1$ but $r^3 \neq 1$.
(b) Find $\mathrm{Gal}(K/F)$ and express each intermediate field between F and K as $F(b)$ for appropriate b in K.