Real Analysis Qualifying Examination

Spring 2014

There are five problems, each worth 20 points. Give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

1. (a) Give an example of a continuous function f in $L^1(\mathbb{R})$ for which $f(x) \nrightarrow 0$ as $|x| \to \infty$.
   (b) Prove that if $f \in L^1(\mathbb{R})$ and uniformly continuous, then $\lim_{|x|\to\infty} f(x) = 0$.

2. Let $\{a_n\}$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} a_n b_n < \infty$ for any given sequence $\{b_n\}$ of non-negative real numbers for which $\sum_{n=1}^{\infty} b_n^2 < \infty$. Prove that $\sum_{n=1}^{\infty} a_n^2 < \infty$.

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$f(x) \geq \limsup_{y\to x} f(y)$$
for all $x \in \mathbb{R}$. Prove that f is Borel measurable.

4. Let (X, μ) be a finite measure space and f be a non-negative measurable function on X. Prove that $\lim_{n\to\infty} \int_X f^n \, d\mu$ is either ∞ or $\mu(f^{-1}(1))$, and characterize the collection of functions f of each type.

5. Let $f, g \in L^1([0,1])$ and for each $0 \leq x \leq 1$ define
$$F(x) := \int_0^x f(y)\,dy \quad \text{and} \quad G(x) := \int_0^x g(y)\,dy.$$
Prove that
$$\int_0^1 F(x)g(x)\,dx = F(1)G(1) - \int_0^1 f(x)G(x)\,dx.$$