1785

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WRITTEN TEST, 25 PROBLEMS / 90 MINUTES / 250 POINTS

October 21, 2023

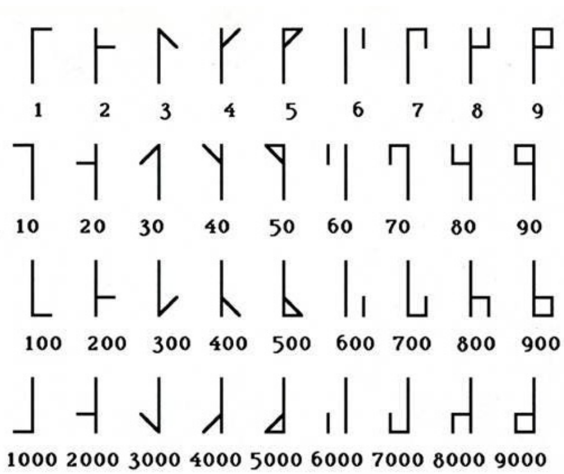
Problem 1. What is the sum of the squares of the digits of the square of the sum of the digits of 2023?

(A) 10 (B) 31 (C) 61 (D) 64 (E) 97

Problem 2. Let C be a circle of radius 5, centered at $(5,3)$. The parabola of equation $y = f(x)$ shares both its x and y intercepts with C. What is $f(2)$?

(A) $-\frac{7}{3}$ (B) $-\frac{5}{3}$ (C) -2 (D) $-\frac{2}{3}$ (E) $-\frac{2}{3}$

Problem 3. During the Middle Ages, Cistercian monks developed an interesting additive numeration system where each number from 1 to 9999 could be expressed as a single symbol. Their convention is illustrated in the table below:



For example, 2023 would be represented by , 1453 by  and 732 by . If X is the largest multiple of 4 whose Cistercian notation is invariant under a 180^o rotation, what is X?

(A) 6996 (B) 8080 (C) 8008 (D) 8888 (E) 9696

Problem 4. Amongst four friends, Alice, Bob, Charly and Donna, each person either always lies or always tell the truth. One evening, they make the following statements:

Alice - Bob is a liar!
Charly - Alice is a liar.
Donna - Alice and Charly are both liars. Bob is a liar!

Who are the liars?

(A) Alice & Bob (B) Alice & Charlie (C) Alice & Donna
(D) Bob & Donna (E) Charlie & Donna

Problem 5. Alice and Bob are bored and decide to play a game. The players alternate taking turns and add 1 or 2 (to their liking) to the number that the previous player has given. The first player who says the number "n" wins. If both players play with perfect strategy, which of the following n ensures that "Alice wins the game?

(A) 26 (B) 30 (C) 34 (D) 38 (E) 42

Problem 6. An ant starts on one corner of a cube, and randomly chooses an edge of the cube to walk across. After reaching the next corner, the ant once again chooses one of the three available edges to walk across. If the ant continues in this fashion, after walking across edges of the cube 6 times, what is the probability of the ant ending on the vertex where it started?

(A) $\frac{61}{243}$ (B) $\frac{17}{81}$ (C) $\frac{1}{8}$ (D) $\frac{17}{64}$ (E) $\frac{41}{162}$

Problem 7. Starting with a regular 2023-gon, suppose that you choose some number of pairs of vertices and draw the diagonals between them, ensuring that the diagonals do not intersect. What is the least number of diagonals you can draw so that it is not possible to include any additional non-intersecting diagonals?

(A) 1612 (B) 1774 (C) 1922 (D) 1993 (E) 2020

Problem 8. Let x,y and z be real numbers and assume that $\frac{xyz}{y+z} = -1$, $\frac{xyz}{x+z} = 1$ and $\frac{xyz}{x+y} = 2$; which of the following could be the value of xyz?

(A) $-\frac{8}{\sqrt{5}}$ (B) $-\frac{4}{5\sqrt{3}}$ (C) $-\frac{1}{\sqrt{2}}$ (D) $\frac{5}{2\sqrt{7}}$ (E) $\frac{3}{\sqrt{2}}$

Problem 9. On planet Zglub, everything is made out of 4 fundamental particles: the archon $\bigcap$, the dawgon Δ, the bullon $\bigotimes$ and the touchdon $\bigsqcup$. These particles are very unstable and after each collision the total number of particles decreases by 1 according to the following rules :

1. An archon always gets absorbed : $\bigcap + X \to X$, where X is any of the four particles.
2. Two identical particles transform into an archon : $X + X \to \bigcap$.
3. A touchdon transforms a dawgon into a bullon, and transforms a bullon into a dawgon : $\Delta + \bigsqcup \to \bigotimes$ and $\bigotimes + \bigsqcup \to \Delta$.
4. A dawgon and a bullon give a touchdon $\Delta + \bigotimes \to \bigsqcup$.

Inside his lab, a scientist has created a small scale model of the planet by putting together 441 archons ($\bigcap$), 673 dawgons (Δ), 431 bullons ($\bigotimes$) and 478 touchdons ($\bigsqcup$). After 2022 collisions, only one particle survives the experiment. What is that terminal particle?

(A) $\bigotimes$ (B) Δ (C) $\bigcap$ (D) $\bigsqcup$

Problem 10. Moody the donkey is very stubborn. He only moves eastwards (E) or northwards (N). Every morning, he leaves from his stable and moves 1/2 mile either (E) or (N), then he reassesses the situation and walks 1/4 mile either (E) or (N), at that point, he decides to move 1/8th of mile either (E) or (N), etc. That day, Moody decides to alternate, he goes first (E), then (N), then (E), etc. How far from home will he end up?

(A) $\frac{\sqrt{2}}{3}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{\sqrt{5}}{3}$ (D) 1 (E) 2

Problem 11. Today, Moody the donkey (see Problem 10) does not feel adventurous. He picks a path so as to stay as close as possible to home. How far from home (in miles) will he land?

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}}{3}$ (C) $\frac{\sqrt{2}}{3}$ (D) $\frac{\sqrt{3}}{2}$ (E) $\frac{\sqrt{5}}{3}$

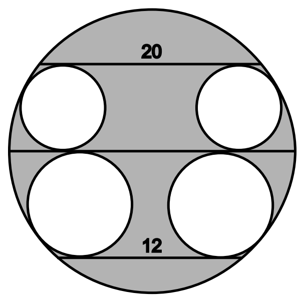
Problem 12. . Arrange the integers in a pyramidal form as follows:

$$
\begin{array}{c}
1 \\
2\ 3 \\
4\ 5\ 6 \\
7\ 8\ 9\ 10 \\
11\ 12\ 13\ 14\ 15
\end{array}
$$

The list goes on with every time n entries on row n. What is the sum of the entries on the 10th row?

(A) 300 (B) 369 (C) 505 (D) 561 (E) 565

Problem 13. Consider the figure below representing a configuration of two chords parallel to a same diameter and four circles, pairwise tangent to the diagonal and one of the two chords. If the length of the chords are as above, what is the area of the shaded region?



(A) 34π (B) 36π (C) 64π (D) 68π (E) 100π

Problem 14. Evaluate the sum $\displaystyle\sum_{n=-3}^{4} \log(n + \sqrt{n^2 + 9})$, i.e.

$$\log(-3 + \sqrt{3^2 + 9}) + \log(-2 + \sqrt{2^2 + 9}) + \cdots + \log(4 + \sqrt{4^2 + 9}).$$

(A) 0 (B) $2\log(3)$ (C) $3\log(3)$ (D) $8\log(3)$ (E) $9\log(3)$

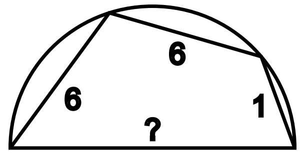
Problem 15. The distinct real numbers a, b and c all satisfy the same equality :

$$a = \sqrt[3]{37a + 84}, \quad b = \sqrt[3]{37b + 84}, \quad c = \sqrt[3]{37c + 84}.$$

What is the sum of the digits of their product, abc?

(A) 3 (B) 12 (C) 5 (D) 6 (E) 7

Problem 16. What is the diameter of the semicircle below given that its endpoints can be joined by three connected chords whose lengths are 6, 6 and 1, as shown?



(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Problem 17. Let f be a real function satisfying the identity

$$f(x - 2023f(y)) = 1 - x - y$$

for all real numbers x and y. What is $f(-1)$?

(A) $\frac{2025}{2024}$ (B) $\frac{2024}{2023}$ (C) $\frac{2023}{2022}$ (D) $-\frac{2023}{2022}$ (E) 1 (F) There is no such function

Problem 18. In a right triangle with sides of length a, b and c, we can compute the ratio

$$\rho = \frac{a+b+c}{c}$$

where c is the longest side. Which of these numbers is a possible value for ρ?

(A) 1.5 (B) 1.9 (C) 2.3 (D) 2.8 (E) 3.2

Problem 19. Let $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ where $i = \sqrt{-1}$. Evaluate the following sum:

$$\sum_{a,b,c=1}^{30} z^{a^2b+bc^2-169b}.$$

(A) 12000 (B) $300 - 600i$ (C) $9000i$ (D) 9000 (E) $6000 + 3000i$

Problem 20. . Let x, y be positive integers less than 100. How many (x, y) pairs satisfies the following equation?

$$2023x^2 = 2025y - 4$$

(A) 0 (B) 1 (C) 4 (D) 100 (E) 400

Problem 21. Consider the list of all integers $n, m > 0$ such that

$$1! + 2! + 3! + \cdots + m! = n^2.$$

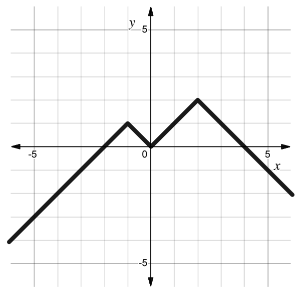
What is the product of all the elements in that list?

(A) 1 (B) 9 (C) 135 (D) 225 (E) 2025

Problem 22. . How many pairs of integers (m, n), where $m, n \in 1, 2, 3, \ldots$ are such that $n^3 + m^3$ divides $n^2 + 6nm + m^2$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 23. The continuous piecewise linear function f is depicted below. The graph has 3 corners. How many corners does $(f \circ f)(x) = f(f(x))$ have?



(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 24. . Consider the set $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. We want to cover S with 5 closed intervals of equal length. What is the minimal length of each individual interval?

(A) $\frac{1}{10}$ (B) $\frac{3}{25}$ (C) $\frac{1}{8}$ (D) $\frac{1}{6}$ (E) $\frac{1}{5}$

Problem 25. In the equation below, a, b, c, and d are base-10 digits. Moreover, assume that neither a nor c equals 0. What is cd if

$$abcd = (ab)^2 + (cd)^2?$$

Here, $abcd$ is a 4-digit number, and ab and cd are 2-digit numbers.

(A) 30 (B) 31 (C) 32 (D) 33 (E) 34

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