Problem 1. If $a$ and $b$ are real numbers whose average is 10, what is the average of $a$, $b$, and 16?

Answer. 12

Solution. Since $\frac{a+b}{2} = 10$, $a + b = 20$, and so $a + b + c = 36$. Dividing by 3 shows that the average is 12.

Problem 2. How many solutions to $x + y = z$ are there if $x, y, z$ are (not necessarily distinct) elements of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$? Note that $1+2=3$ and $2+1=3$ are different solutions.

Answer. 45

Solution. Given $x$, the integer $y$ must be one of the $10-x$ numbers $1, 2, 3, \ldots, 10-x$, and then $z$ is determined as $x+y$. So there are $\sum_{x=1}^{10}(10-x) = \sum_{x=0}^{9} x = \frac{9 \cdot 10}{2} = 45$ solutions.

Problem 3. Find a value $x > 1$ so that a 1 by $x$ rectangle can be cut into two congruent rectangles each similar to the original 1 by $x$ rectangle.
Answer. \( \sqrt{2} \) (or \( x = \sqrt{2} \))

Solution. Cut the rectangle in half vertically:

\[ \begin{array}{c}
\text{1} \\
\text{x/2} \\
\text{x/2}
\end{array} \]

The similarity condition then requires \( \frac{x/2}{1} = \frac{1}{x} \), so \( x = \sqrt{2} \).

Problem 4. How many subsets of \( \{U, G, A, H, S, M, T\} \) have nonempty intersection with \( \{U, G, A\} \)?

Answer. 112 (sets)

Solution. Our set must be one of the \( 2^7 = 128 \) subsets of \( \{U, G, A, H, S, M, T\} \), but not one of the \( 2^4 = 16 \) subsets of \( \{H, S, M, T\} \). This leaves \( 128 - 16 = 112 \) possibilities.

Problem 5. I got the following message on my phone: “Your screen time was down 37% from last week for an average of 9 minutes/day.” What was my screen time, in minutes, last week?

Answer. 100 (minutes)

Solution. This week my screen time was (only!) \( 9 \times 7 = 63 \) minutes. Solving \( (1 - .37)x = 63 \),

gives \( x = 100 \), so last week I had 100 minutes of screen time.

Problem 6. Let \( L \) be the line segment in \( \mathbb{R}^2 \) from \( (0,0) \) to \( (2,0) \). At each point \( (x,0) \) of \( L \) draw a disk of radius 1 centered at \( (x,0) \). What is the area of the union of these disks? (A disk consists of a circle together with the points inside the circle.)

Answer. \( 4 + \pi \)

Solution. Here is a picture of the union of disks:
Notice that it consists of a $2 \times 2$ rectangle and 2 half-disks of radius 1, so the area is $4 + \pi$.

**Problem 7.** What is $\arcsin \left( \frac{\pi}{4} \right) + \arccos \left( \frac{\pi}{4} \right)$?

**Answer.** $\frac{\pi}{2}$

**Solution.** Let $\alpha$ and $\beta$ be the angles in the following triangle:

Then $\alpha = \arcsin \left( \frac{a}{b} \right)$ while $\beta = \arccos \left( \frac{a}{b} \right)$. Since $\alpha + \beta = \frac{\pi}{2}$ for any $0 < a < b$, we find that actually $\arcsin (x) + \arccos (x) = \frac{\pi}{2}$ for any $-1 \leq x \leq 1$.

**Problem 8.** What is the largest positive integer $k$ for which $3^k$ divides $\underbrace{999\ldots9}_{2019 \text{ nines}}$?

**Answer.** 3 (or $k = 3$)

**Solution.** Let $N = \underbrace{999\ldots9}_{2019 \text{ nines}}$. Clearly, $3^2 = 9 \mid N$, and $N/3^2 = 111\ldots1$. The sum of the digits of $N/3^2$ is 2019, which is a multiple of 3 but not of 9. So by the familiar divisibility rules for 3 and 9, we see that $N/3^2$ is a multiple of 3 but not of 9. Thus, $3^3 \mid N$, while $3^4 \not\mid N$.

**Problem 9.** Detective Uga comes across a combination lock requiring a 5-digit key-code, with each digit in $\{0, 1, 2, \ldots, 9\}$. Dusting for prints reveals that the combination only uses the digits 2, 0, 1, 9 and only those four digits. How many possible
keycodes use those four digits?

**Answer.** 240 (combinations)

**Solution.** Exactly one of the digits 2,0,1,9 occurs twice. There are 4 choices for which one. Having made that choice, there are \( \frac{5!}{2} = 60 \) corresponding ways to permute the digits; here the repeated digit accounts for the division by 2. Thus, there are \( 4 \cdot 60 = 240 \) combinations in total.

**Problem 10.** If \( p(x) \) is a degree 2 polynomial with positive integer coefficients, with \( p(1) = 11 \) and \( p(10) = 236 \), what is \( p(-1) \)?

**Answer.** 5

**Solution.** Write \( p(x) = ax^2 + bx + c \) with \( a, b, c \) positive integers. Then \( 11 = p(1) = a + b + c \), and so \( a = 11 - b - c \leq 11 - 1 - 1 = 9 \). Similarly, \( b, c \leq 9 \). Since \( a, b, c \in \{1, 2, \ldots, 9\} \), we see that \( p(10) = 100a + 10b + c \) has decimal expansion \( abc \). So \( a = 2, b = 3, c = 6 \), and \( p(-1) = a(-1)^2 + b(-1) + c = a - b + c = 5 \).

**Authors.** Written by Mo Hendon, Paul Pollack, and Peter Woolfitt.