Sponsored by: UGA Math Department and UGA Math Club

Ciphering Round / 2 minutes per problem
October 26, 2019

WITH SOLUTIONS

Problem 1. If a and b are real numbers whose average is 10, what is the average of a, b, and 16?

Answer. 12

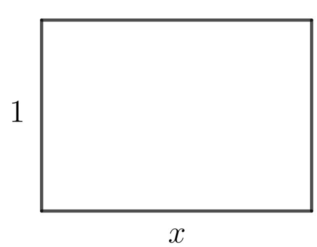
Solution. Since $\frac{a+b}{2} = 10$, $a + b = 20$, and so $a + b + c = 36$. Dividing by 3 shows that the average is 12.

Problem 2. How many solutions to $x+y = z$ are there if x, y, z are (not necessarily distinct) elements of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$? Note that $1+2 = 3$ and $2+1 = 3$ are different solutions.

Answer. 45

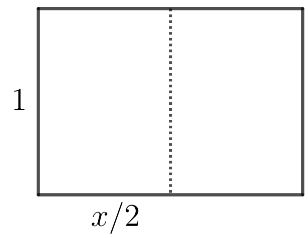
Solution. Given x, the integer y must be one of the $10-x$ numbers $1, 2, 3, \ldots, 10-x$, and then z is determined as $x + y$. So there are $\sum_{x=1}^{10}(10 - x) = \sum_{x=0}^{9} x = \frac{9\cdot 10}{2} = 45$ solutions.

Problem 3. Find a value $x > 1$ so that a 1 by x rectangle can be cut into two congruent rectangles each similar to the original 1 by x rectangle.



Answer. $\sqrt{2}$ (or $x = \sqrt{2}$)

Solution. Cut the rectangle in half vertically:



The similarity condition then requires $\dfrac{x/2}{1} = \dfrac{1}{x}$, so $x = \sqrt{2}$.

Problem 4. How many subsets of $\{U, G, A, H, S, M, T\}$ have nonempty intersection with $\{U, G, A\}$?

Answer. 112 (sets)

Solution. Our set must be one of the $2^7 = 128$ subsets of $\{U, G, A, H, S, M, T\}$, but *not* one of the $2^4 = 16$ subsets of $\{H, S, M, T\}$. This leaves $128 - 16 = 112$ possibilities.

Problem 5. I got the following message on my phone: "Your screen time was down 37% from last week for an average of 9 minutes/day." What was my screen time, in minutes, last week?

Answer. 100 (minutes)

Solution. This week my screen time was (only!) $9 \times 7 = 63$ minutes. Solving

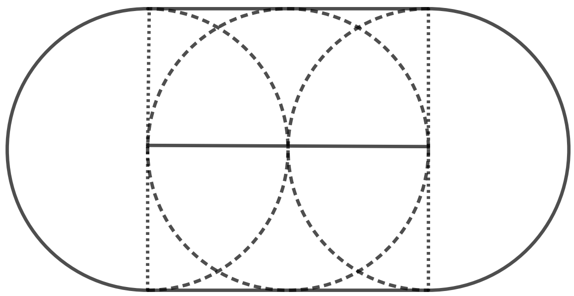
$$(1 - .37)x = 63,$$

gives $x = 100$, so last week I had 100 minutes of screen time.

Problem 6. Let L be the line segment in $\mathbb{R}^2$ from $(0, 0)$ to $(2, 0)$. At each point $(x, 0)$ of L draw a disk of radius 1 centered at $(x, 0)$. What is the area of the union of these disks? (A *disk* consists of a circle together with the points inside the circle.)

Answer. $4 + \pi$

Solution. Here is a picture of the union of disks:

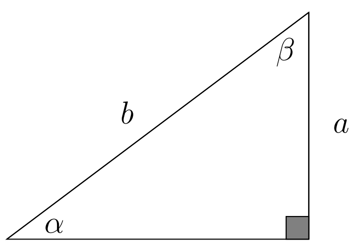


Notice that it consists of a 2×2 rectangle and 2 half-disks of radius 1, so the area is $4 + \pi$.

Problem 7. What is $\arcsin\left(\frac{\pi}{4}\right) + \arccos\left(\frac{\pi}{4}\right)$?

Answer. $\frac{\pi}{2}$

Solution. Let α and β be the angles in the following triangle:



Then $\alpha = \arcsin\left(\frac{a}{b}\right)$ while $\beta = \arccos\left(\frac{a}{b}\right)$. Since $\alpha + \beta = \frac{\pi}{2}$ for any $0 < a < b$, we find that actually $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$ for any $-1 \leq x \leq 1$.

Problem 8. What is the largest positive integer k for which 3^k divides $\underbrace{999\ldots9}_{\text{2019 nines}}$?

Answer. 3 (or $k = 3$)

Solution. Let $N = \overbrace{999\ldots9}^{\text{2019 nines}}$. Clearly, $3^2 = 9 \mid N$, and $N/3^2 = 111\ldots1$. The sum of the digits of $N/3^2$ is 2019, which is a multiple of 3 but not of 9. So by the familiar divisibility rules for 3 and 9, we see that $N/3^2$ is a multiple of 3 but not of 9. Thus, $3^3 \mid N$, while $3^4 \nmid N$.

Problem 9. Detective Uga comes across a combination lock requiring a 5-digit key-code, with each digit in $\{0, 1, 2, \ldots, 9\}$. Dusting for prints reveals that the combination only uses the digits $2, 0, 1, 9$ and only those four digits. How many possible keycodes use those four digits?

Answer. 240 (combinations) **or** $4^5 = 1024$ (combinations)

Solution. Exactly one of the digits 2,0,1,9 occurs twice. There are 4 choices for which one. Having made that choice, there are $\frac{5!}{2} = 60$ corresponding ways to permute the digits; here the repeated digit accounts for the division by 2. Thus, there are $4 \cdot 60 = 240$ combinations in total.

Alternative solution. The above solution gives the intended answer. However, the wording of the problem was imprecise. It should have specified that the combination uses *precisely* the four digits 2, 0, 1, 9. As it stands, the problem could be interpreted to allow any subset of $\{2, 0, 1, 9\}$ as the digits of the combination. In that case, there are 4 choices for each of the 5 digits, and so $4^5 = 1024$ possible combinations.

Problem 10. If $p(x)$ is a degree 2 polynomial with positive integer coefficents, with $p(1) = 11$ and $p(10) = 236$, what is $p(-1)$?

Answer. 5

Solution. Write $p(x) = ax^2 + bx + c$ with a, b, c positive integers. Then $11 = p(1) = a + b + c$, and so $a = 11 - b - c \leq 11 - 1 - 1 = 9$. Similarly, $b, c \leq 9$. Since $a, b, c \in \{1, 2, \ldots, 9\}$, we see that $p(10) = 100a + 10b + c$ has decimal expansion abc. So $a = 2, b = 3, c = 6$, and $p(-1) = a(-1)^2 + b(-1) + c = a - b + c = 5$.

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