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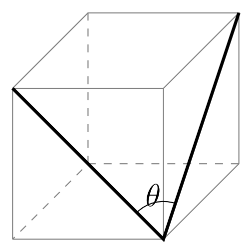
Ciphering Round / 2 minutes per problem
October 22, 2022

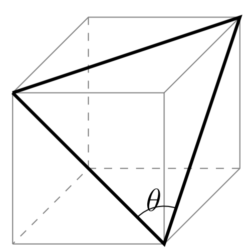
WITH SOLUTIONS

Problem 1. What is the angle, in degrees, between two face diagonals of a cube meeting at the same corner?



Answer. $60°$

Solution. Notice that adding a third face diagonal creates an equilateral triangle, so the angle must be $60°$.



Problem 2. Suppose x is a single digit $0, 1, \ldots, 9$. If the 8 digit number $23456x89$ is divisible by 9, what is x?

Answer. 8

Solution. We'll use the useful fact that in base 10 a number is divisible by 9 if and only if the sum of its digits is divisible by 9, so 9 needs to divide

$$2 + 3 + 4 + 5 + 6 + x + 8 + 9 = 37 + x,$$

so we need $x = 8$.

Problem 3. If you reflect a line of slope π over the y-axis, what is the slope of the reflected line?

Answer. $-\pi$

Solution. Suppose (x_0, y_0) and (x_1, y_1) are on the original line. Then $(-x_0, y_0)$ and $(-x_1, y_1)$ are on the reflected line. The slope of the reflected line is then

$$\frac{y_1 - y_0}{-x_1 - (-x_0)} = -\frac{y_1 - y_0}{x_1 - x_0} = -\pi.$$

Problem 4. Suppose you have 10 distinct numbers. The average of the smallest 4 numbers is 20, while the average of the largest 6 numbers is 30. What is the average of all 10 numbers?

Answer. 26

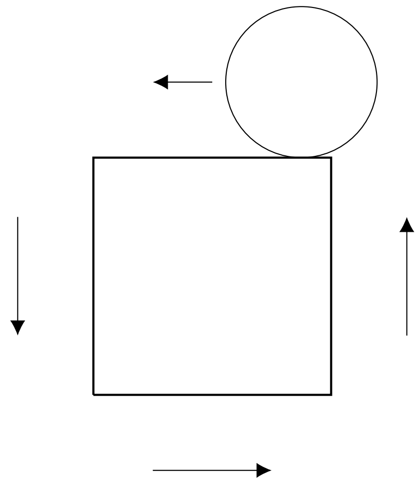
Solution. Since the average of the smallest 4 numbers is 20, their sum is 80. Similarly, the sum of the largest 6 numbers is 180. So the sum of all ten numbers is 260, and their average is 26.

Problem 5. Red stones weigh 1 pound while black stones weigh 2 pounds. How many different combinations of red and black stones weigh a total of 100 pounds? A combination must contain a minimum of one red stone and one black stone.

Answer. 49

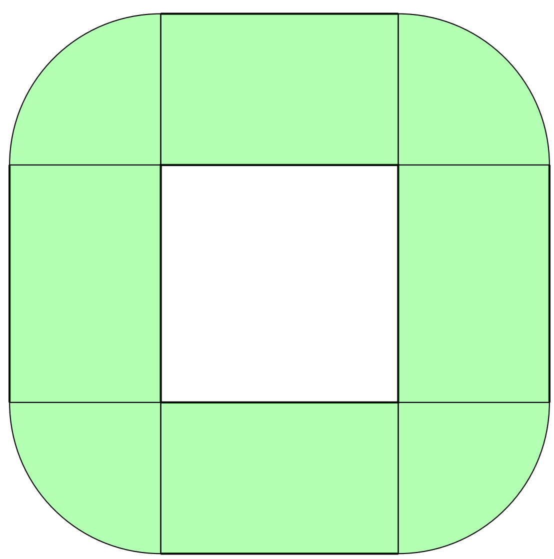
Solution. We only need consider how many black stones we use, then fill out the rest using red stones to get to 100 pounds. Since there must be at least one red stone, there can be anywhere from 1 to 49 black stones, yielding a total of 49 combinations.

Problem 6. You have a square of sidelength π and a disk of radius 1. The disk rolls all the way around the outside of the square, and always remains tangent (or touching the square only at the corner). What is the area of the path traveled by the disk?



Answer. 12π

Solution. Notice that the region can be divided into 4 rectangles and 4 quarter circles. The widths of the rectangles are 2, and their lengths add up to 4π, the perimeter of the square, so they contribute 8π to the area. The 4 quarter circles have radius 2, so total area $\pi(2)^2 = 4\pi$. Thus the total area is 12π.



Problem 7. Find the maximum possible value of the function

$$1 + \sin^2(x) + \sin^4(x) + \cdots + \sin^{2022}(x) + \cos^{2022}(x) + \cdots + \cos^4(x) + \cos^2(x) + 1.$$

[Here the sum is over all the even powers of $\sin(x)$ and $\cos(x)$ from 0 to 2022.]

Answer. 1013

Solution. Since $|\sin(x)|$ and $|\cos(x)|$ are bounded by 1, we have $\sin^{2k}(x) \leq \sin^2(x)$ and $\cos^{2k}(x) \leq \cos^2(x)$ for $k \geq 1$. Thus the given sum is bounded by

$$1 + 1011\sin^2(x) + 1011\cos^2(x) + 1 = 1013.$$

Plugging in say $x = 0$ shows that this bound is attainable.

Problem 8. Fully simplify

$$4^{4/\log_3(4)}$$

Answer. 81

Solution. We'll use the change of base formula: For $a, b > 1$, $\log_a(b) = \frac{1}{\log_b(a)}$. Thus

$$4^{4/\log_3(4)} = 4^{4\log_4(3)} = 4^{\log_4(3^4)} = 3^4 = 81$$

So why is the change of base formula true? Notice that it can be rewritten

$$\log_a(b)\log_b(a) = 1.$$

To check this, compute

$$a^{\log_a(b)\log_b(a)} = \left(a^{\log_a(b)}\right)^{\log_b(a)} = b^{\log_b(a)} = a.$$

Problem 9. Suppose the nonzero real numbers a and b are the roots of $x^2 + ax + b$. Find a and b.

Answer. $a = 1$, $b = -2$

Solution. Since a and b are roots, we can factor $x^2 + ax + b$ as

$$x^2 + ax + b = (x - a)(x - b) = x^2 - (a + b)x + ab.$$

Matching up coefficients, we get the equations $a = -(a + b)$ and $b = ab$. Since $a, b \neq 0$, we get $a = 1$ and $b = -2$.

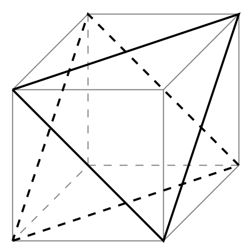
Notice that we have proved: For a monic (leading coefficient is 1) quadratic polynomial, the constant term is the product of the roots, while the coefficient of the linear term is the negative sum of the roots. What is the corresponding theorem for monic cubic polynomials? For higher degree?

Problem 10. How many equilateral triangles can be formed by joining the vertices of a cube?

Answer. 8

Solution. Choose any corner of the cube. It is adjacent to 3 other corners, and those form the vertices of an equilateral triangle, so there's one equilateral triangle for each corner of the cube.

Sometimes people think that the two triangles arising from opposite corners of the cube are the same, but this is not true. In fact, two thirds of the volume of the cube is between those two triangles. It's an interesting exercise to try to visualize the region between those two triangles. Try it!



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