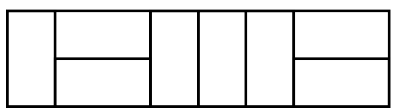
Sponsored by the UGA Math Department

CIPHERING ROUND, 10 PROBLEMS / 2 MINUTES PER PROBLEM / 100 POINTS

October 21, 2023

WITH SOLUTIONS

Problem 1. A $2 \times N$ room is tiled with small rectangles of size 1×2. An example of tiling is given in the picture below. If $N = 8$, in how many ways can you tile the room?



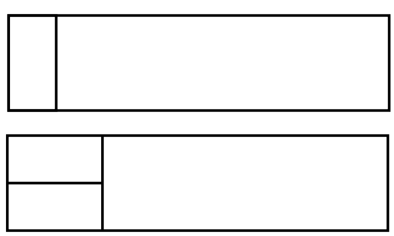
Answer. 34

Solution. Call T_N the number of ways of tiling a $2 \times N$ room. One can observe that the sequence T_N obeys a Fibonacci like recurrence:

$$T_{N+2} = T_{N+1} + T_N.$$

That is, all possible tilings of a $2 \times (N+2)$ room can be decomposod as either:

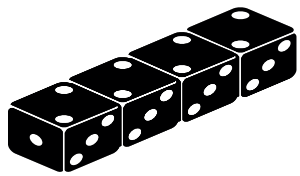
1. Two tiles placed horizontally and a tiling of a $2 \times N$ room.
2. One tile placed vertically and a tiling of a $2 \times (N+1)$



Finally, noting that $T_1 = 1$ while $T_2 = 2$ allows us to iterate the recursion for the required number of tilings:

$$T_1 = 1, T_2 = 2, T_3 = 3, T_4 = 5, T_5 = 8, T_6 = 13, T_7 = 21, T_8 = 34.$$

Problem 2. Four identical dice are stacked like on the picture below. The internal dice show four faces and the boundary ones five. What is the sum of the dots on the 18 visible faces? Recall that on a dice, opposite sides add up to 7.



Answer. 63

Solution. As opposite sides add up to seven, the visible sides of the two central dice add up to $2 \times ((1+2+\cdots+6)-7) = 2 \times (21-7) = 28$. The first and the last dice being identical, we have again lost sight of $7 = 1+6$ pips. The total number of visible dots on the lateral dice is $2 \times 21 - 7 = 35$. All in all, $35 + 28 = 63$ pips are visible.

Problem 3. Two points in the plane have integer coordinates and the distance between them is $\sqrt{65}$. If these points form the opposite corners of a rectangle whose sides are parallel to the coordinate axes, what is the maximal area of such a rectangle.

Answer. 28

Solution. Without loss of generality, one of the points can be chosen to be the origin $(0,0)$ and the other point can be taken in the first quadrant with coordinates (a,b) where $a^2 + b^2 = 65$. Now 65 can be written as the sum of two squares in two non equivalent ways : $1^2 + 8^2$ and $4^2 + 7^2$. The area, $4 \times 7 = 28$, will be maximal in the latter case.

Problem 4. What is the remainder of the following integer division?

$$\frac{1000^4 + 1001^4 + 1002^4 + \cdots + 2023^4}{16}$$

Answer. 0

Solution. Any even number to the 4th power is divisible by 16 since $(2n)^4 = 16n^4$. Also, any odd number to the 4th power has a remainder of 1 after divison by 16 since $(2n+1)^4 = 16n^4 + 4(8n^3) + 6(4n^2) + 4(2n) + 1 = 16n^4 + 32n^3 + 8n(3n+1) + 1$ and $8n(3n+1)$ is always multiple of 16. Since we have $\dfrac{2023-1001}{2} + 1 = 512$ odd numbers, the remainder will be the remainder of $\dfrac{512}{16}$ which is 0.

Problem 5. In the expression below, a,b and c represent digits. If

$$0.2a \times 7.b = 2.c,$$

what is $a \times b \times c + a + b + c$?

Answer. 54

Solution. As the resulting number has only one digit after the period, ab must be a multiple of 10 hence either a or b is 0 or a or b equals 5.
If a or b is 0, it is easy to see that the product can never be of the form $2.c$. If $a = 5$, $0.25 \times 7.b < 2$ hence this won't work. The last option is $b = 5$ and a even. In which case, only 0.28 is large enough. The above equality is thus $0.28 \times 7.5 = 2.1$. We conclude with $8 \times 5 \times 1 + 1 + 5 + 8 = 54$.

Problem 6. What is the last digit of M?

$$M = 1! + 2! + 3! + \cdots + 2023!$$

Answer. 3

Solution. Computing the first few terms we get $1! + 2! + 3! + 4! = 33$. Now $5! = 120$ ends with 0 and so do all subsequent terms as they are multiples of $5!$. The last digit of M is thus 3.

Problem 7. Find the integer a that satisfies

$$a!(a+1)! = 10!$$

Answer. 6

Solution. Note that the prime decomposition of 10! contains a unique copy of 7. We must thus have $a = 6$.

Problem 8. How many triangles have sides of length π, $\sqrt{2}$ and a where the latter is an integer? (the order of the sides is irrelevant)

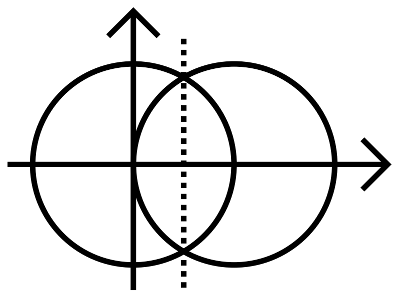
Answer. 3

Solution. First note that $\pi + \sqrt{2} = 4.55...$ and $\pi - \sqrt{2} = 1.72....$ By the triangle inequality, we can thus frame $a : 1.72.. < a < 4.55....$ This being the only constraint, we have $a \in 2, 3, 4$ and there are 3 triangles.

Problem 9. Everytime a frog jumps, it makes a 1 foot leap in an arbitrary direction in the plane. What is the probability that after 2 leaps, the frog is within 1 foot from its starting position?

Answer. $\frac{1}{3}$

Solution. Without loss of generality, we can assume that the frog starts at the origin of the coordinate plane. After one jump, it lies on the unit circle, $x^2 + y^2 = 1$, but up to a change of coordinates, we can assume that point to be $(1, 0)$.



After the second jump, the frog will lie on the circle $(x - 1)^2 + y^2 = 1$. By symmetry, the proportion of the second circle within the first one is exactly $\frac{1}{3}$: the two circles meet along the $x = \frac{1}{2}$ line hence cut an arc of amplitude $2 \times 60^o = 120^o$, that is 1/3rd.

Problem 10. If the integers a, b, c and d are such that

$$2^a 3^b 5^c 7^d - 77 = 2023,$$

what is $a + b + c + d$?

Answer. 6

Solution. $2023+77 = 2100$ can be factored as $3^1 \times 7^1 \times 2^2 \times 5^2$ hence $a+b+c+d = 6$.

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