1. If \( \{X_n\} \) are iid random variables with \( P(X_1 = 0) < 1 \) and \( S_n = X_1 + X_2 + \cdots + X_n \), then for every \( c > 0 \), there exists an integer \( n_0 \) such that \( P(|S_n| > c) > 0 \).

2. (a) Given a random variable \( X \) with finite mean square. Let \( \mathcal{D} \) be a \( \sigma \)-algebra. Show that \( E[X|\mathcal{D}] \) is the minimizer of \( E(X - \xi)^2 \) over all \( \mathcal{D} \)-measurable r.v.s \( \xi \), i.e.,

\[
E(X - E[X|\mathcal{D}])^2 \leq E(X - \xi)^2
\]

for all \( \mathcal{D} \)-measurable r.v.s \( \xi \).

(b) Let \((\Omega, \mathcal{F}, P)\) denote a probability space. Suppose \( f : \mathbb{R}^n \times \Omega \to \mathbb{R} \) is a bounded \( \mathcal{B}(\mathbb{R}^n) \times \mathcal{C} \) measurable function and \( X \) be a \( n \)-dimensional \( \mathcal{D} \) measurable random variable. Assume \( \mathcal{C} \) and \( \mathcal{D} \) are independent. If \( g(x) := E[f(x, \omega)] \), then

\[
g(X) = E[f(X, \omega)|\mathcal{D}], \text{ a.s.}
\]

3. Show that random variables \( X_n, n \geq 1 \), and \( X \) satisfy \( X_n \to X \) in distribution iff

\[
E[F(X_n)] \to E[F(X)]
\]

for every continuous distribution function \( F \).

4. Let \( \{X_n\} \) be a sequence of iid random variables with \( E|X_1| = \infty \). Let \( S_n = X_1 + X_2 + \cdots + X_n \). Show that

\[
P \left( \limsup_n \frac{|S_n|}{n} = \infty \right) = 1.
\]

5. Let \( \{X_n\} \) be a sequence of iid random variables with \( EX_1 = 0 \). Prove that (a) the sequence \( \{\frac{S_n}{n}\} \) is uniformly integrable; (b) \( \frac{|S_n|}{n} \to 0 \).

6. Let \( \{X_n\} \) be iid r.v.s with distribution \( F(x) \) having finite mean \( \mu \) and variance \( \sigma^2 > 0 \). Let \( S_n = X_1 + \cdots + X_n \). Show that

\[
\frac{S_n - n\mu}{\sigma\sqrt{n}} \to N(0,1) \text{ in distribution as } n \to \infty.
\]

Here \( N(0,1) \) is a standard normal random variable.

7. Let \( X_1, X_2, \ldots \) be a sequence of independent r.v.s with \( EX_i = 0 \). Let \( S_n = X_1 + X_2 + \cdots + X_n \) and \( \mathcal{F}_n = \sigma\{X_1, \ldots, X_n\} \). Show that \( \phi(S_n) \) is an \( \mathcal{F}_n \)-submartingale for any convex \( \phi \) provided that \( E[|\phi(S_n)|] < \infty \) for all \( n \).