# Real Analysis Preliminary Examination — September 1996

Work three problems from each Section. All functions are real–valued. Unless specified otherwise, integrals are to be taken with respect to Lebesgue measure, denoted $m$.

## Section A

1. Suppose $A$ and $B$ and non–empty subsets of $\mathbb{R}$ satisfying $x \le y$ for all $x \in A$ and $y \in B$. Prove that $\sup A \le \inf B$.

2. Suppose the sequence $\{f_n\}_{n\in\mathbb{N}}$ in $C[0,1]$ converges uniformly to a function $g$. Prove that the family $\{f_n : n \in \mathbb{N}\}$ is equicontinuous.

3. State and prove a version of the chain rule for functions mapping $\mathbb{R}^n$ into itself.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded, but not necessarily measurable and define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \limsup_{y\to x} f(y)$. Prove that $g$ is Lebesgue measurable.

## Section B

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable with $\int (1+x^2)|f(x)|dm(x) < \infty$. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(t) = \int f(x) ln(1+t^2x^2)dm(x)$. Prove that $F$ is well–defined and differentiable and find a formula for $F'$.

2. Prove that if $f, g \in L^2(m)$, then their convolution $f * g$ is uniformly continuous on $\mathbb{R}$.

3. Suppose $\mu$ and $\nu$ are finite measures on the same measurable space $(X, \mathcal{B})$. Prove that the following are equivalent.

   (i) $\nu$ is absolutely continuous with respect to $\mu$.

   (ii) For each $\epsilon > 0$ there is a $\delta > 0$ such that $\nu(E) < \epsilon$ whenever $\mu(E) < \delta$.

4. Equip $X = Y = [0,1]$ with the $\sigma$–algebra of Borel sets and the two measures $\mu = m =$ Lebesgue measure and $\nu =$ counting measure. Prove that $(\mu \times \nu)\Delta = \infty$ where $\Delta = \{(x,y) \in [0,1]\times[0,1] : x = y\}$. Then explain the relevance of this example to the Fubini and Tonelli Theorems.

## Section C

1. Define a linear functional $\phi$ on $C[0,1]$ by $\phi(f) = 3f(1) - 2f(0) + \int_0^1 f dm$. Compute the norm of $\phi$ and justify your answer.

2. Show that if the dual $X^*$ of a Banach space $X$ is separable, then $X$ is also separable.

3. Suppose $M$, $N$ are closed subspaces of a Banach space $X$ with $M + N = X$ and $M \cap N = \emptyset$. Prove that there is a constant $C$ such that $\|x + y\| \ge C\|x\|$ for all $x \in M,\ y \in N$.

4. Suppose $\{f_n\} \subset C[0,1]$ with $\sup \int |f_n| = \infty$. Prove that there is a function $g \in L^\infty[0,1]$ satisfying $\sup \int f_n g dm = \infty$.