REAL ANALYSIS PRELIMINARY EXAMINATION

May 11, 1995

A. Do all problems.

1. If $f: \mathbf{R}^n \to \mathbf{R}$ is differentiable at the point a with differential (total derivative) $df(a)$ show that f has partial derivatives at a. Discuss the relation between the existence of $df(a)$ and the existence of partial derivatives of f at a. Consider these matters for the particular function f which is 0 at the point $a = (0,0)$ and defined by

$$f(x,y) = \frac{xy}{x^2 + y^2}$$

for $(x,y) \neq (0,0)$. Finally, prove or disprove that, in general, differentiability is implied by the existence of all partial derivatives.

2. Let $f_1, f_2, \ldots$ be the sequence of functions defined on $\mathbb{R}$ by

$$f_n(x) = \frac{x}{1 + nx^2}$$

Prove or disprove

(i) The sequence converges uniformly to a differentiable function f.

(ii) The sequence of derivatives is convergent.

(iii) The sequence of derivatives is uniformly convergent.

(iv) The sequence of derivatives converges to f'.

B. Do all problems.

1. If f is a Lebesgue integrable function on $\mathbf{R}$, use the basic theorems on integration to show that

$$\left(\frac{\sin nx}{nx}\right)^2 f(x)$$

is integrable for $n = 1,2,\ldots$, and to investigate the existence and value of

$$\lim_{n\to\infty} \int_{\mathbf{R}} \left(\frac{\sin nx}{nx}\right)^2 f(x)dx.$$

2. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on $\mathbb{R}$. Show that the set of points where the sequence is convergent is measurable.

3. Prove or disprove: The complement in the closed unit interval of an open dense subset has Lebesgue measure 0.

4. Consider the formula

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$$

(i) Show that $\Gamma(0) = \infty$.

(ii) Show that the integrand is an L^1 function for any complex s with $Re(s) > 0$.

C. Do all problems.

1. Let V be a normed linear space, W a dense linear subspace, and f a linear functional on W. Show that f has a continuous linear extension to V and that the extension is necessarily unique iff f is bounded.

2. Let $\mathcal{H}$ be a complex Hilbert space.

(i) Prove the Riesz representation theorem.

(ii) Prove there is a canonical conjugate linear isomorphism of $\mathcal{H}$ onto its dual.

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3. State and prove the closed graph theorem. You may assume the open mapping theorem.